

1. Számítsuk ki a következő határozatlan és határozott integrálokat!

$$\begin{array}{lll} \text{a) } \int x^4 - 3x^2 + 2 dx & \text{b) } \int \sqrt[3]{x^2} dx & \text{c) } \int_0^1 \sqrt{x\sqrt{x\sqrt{x}}} dx \\ \text{d) } \int \frac{x^4 + 2x - 1}{x} dx & \text{e) } \int_1^4 \sqrt{x}(7x^2 + 10x - 3) dx & \end{array}$$

2. A következő integrálok kiszámításához alkalmazzuk a láncszabály megfordítását:
 $\int f(g(x))g'(x) dx = F(g(x)) + C$.

$$\begin{array}{lll} \text{a) } \int_0^1 xe^{x^2} dx & \text{b) } \int \frac{\sin(\ln x)}{x} dx & \text{c) } \int \operatorname{tg} x dx \\ \text{d) } \int \frac{x}{\sqrt{1-x^2}} dx & \text{e) } \int_0^{\frac{\pi}{2}} \cos x \sin^3 x dx & \end{array}$$

3. Az integrandus megfelelő átalakítása után alkalmazzuk a láncszabályból adódó
 $\int f(ax+b) dx = \frac{1}{a}F(ax+b) + C$ összefüggést!

$$\begin{array}{llll} \text{a) } \int \sqrt[4]{2-3x} dx & \text{b) } \int \frac{e^x + 1}{e^{2x}} dx & \text{c) } \int \cos^2 x dx & \text{d) } \int_0^{\frac{\pi}{4}} \cos^4 x dx \\ \text{e) } \int \frac{1}{x^2 + 4} dx & \text{f) } \int \frac{1}{\sqrt{1+3x^2}} dx & \text{g) } \int_{-1}^0 (2x+1)^{10} dx & \text{h) } \int \frac{x^3 - 1}{x+2} dx \end{array}$$

4. Számítsuk ki az alábbi improprius integrálokat!

$$\begin{array}{llll} \text{a) } \int_1^\infty \frac{1}{x^2 + 1} dx & \text{b) } \int_0^1 \frac{1}{\sqrt[3]{x^2}} dx & \text{c) } \int_e^\infty \frac{1}{x \ln x} dx & \text{d) } \int_e^\infty \frac{1}{x(\ln x)^2} dx \end{array}$$

5. Keressük meg azt az $f(x)$ függvényt, amelyre

$$\begin{array}{l} \text{a) } f'(x) = 4x + \sin 2x, \text{ és } f(0) = 0; \\ \text{b) } f''(x) = 6x^2 + \frac{1}{x\sqrt{x}}, f(1) = 0, \text{ és } f'(1) = 2. \end{array}$$

Megoldások

1. a) $\int x^4 - 3x^2 + 2 dx = \frac{1}{5}x^5 - x^3 + 2x + C$
- b) $\int \sqrt[3]{x^2} dx = \int x^{2/3} dx = \frac{3}{5}x^{5/3} + C$
- c) $\int_0^1 \sqrt{x\sqrt{x\sqrt{x}}} dx = \int_0^1 x^{7/8} dx = \left[\frac{8}{15}x^{15/8} \right]_0^1 = \frac{8}{15}$
- d) $\int \frac{x^4 + 2x - 1}{x} dx = \int x^3 + 2 - \frac{1}{x} dx = \frac{1}{4}x^4 + 2x - \ln|x| + C$
- e) $\int_1^4 \sqrt{x}(7x^2 + 10x - 3) dx = \int_1^4 7x^{5/2} + 10x^{3/2} - 3x^{1/2} dx = \left[2x^{7/2} + 4x^{5/2} - 2x^{3/2} \right]_1^4 = 2 \cdot 128 + 4 \cdot 32 - 2 \cdot 8 - 4 = 364$
2. a) $\int_0^1 xe^{x^2} dx = \int_0^1 \frac{1}{2}e^{x^2} 2x dx = \left[\frac{1}{2}e^{x^2} \right]_0^1 = \frac{1}{2}(e - 1)$
- b) $\int \frac{\sin(\ln x)}{x} dx = \int \sin(\ln x) \frac{1}{x} dx = -\cos(\ln x) + C$
- c) $\int \operatorname{tg} x dx = \int \frac{\sin x}{\cos x} dx = \int \frac{-1}{\cos x} \cdot (-\sin x) dx = -\ln|\cos x| + C$
- d) $\int \frac{x}{\sqrt{1-x^2}} dx = \int -\frac{1}{2}(1-x^2)^{-1/2}(-2x) dx = -\frac{1}{2} \cdot 2(1-x^2)^{1/2} + C = -\sqrt{1-x^2} + C$
- e) $\int_0^{\pi/2} \cos x \sin^3 x dx = \left[\frac{1}{4} \sin^4 x \right]_0^{\pi/2} = \frac{1}{4}$
3. a) $\int \sqrt[4]{2-3x} dx = \int (-3x+2)^{1/4} dx = \frac{4}{5} \left(-\frac{1}{3}\right) (-3x+2)^{5/4} + C = -\frac{4}{15}(-3x+2)^{5/4} + C$
- b) $\int \frac{e^x + 1}{e^{2x}} dx = \int e^{-x} + e^{-2x} dx = -e^{-x} - \frac{1}{2}e^{-2x} + C$
- c) $\int \cos^2 x dx = \int \frac{1}{2}(1 + \cos(2x)) dx = \frac{1}{2}x + \frac{1}{4} \sin(2x) + C$
- d) $\int_0^{\pi/4} \cos^4 x dx = \int_0^{\pi/4} \left(\frac{1}{2}(1 + \cos(2x))\right)^2 dx = \int_0^{\pi/4} \frac{1}{4} + \frac{1}{2} \cos(2x) + \frac{1}{4} \cos^2(2x) dx = \int_0^{\pi/4} \frac{1}{4} + \frac{1}{2} \cos(2x) + \frac{1}{8}(1 + \cos(4x)) dx = \int_0^{\pi/4} \frac{3}{8} + \frac{1}{2} \cos(2x) + \frac{1}{8} \cos(4x) dx = \left[\frac{3}{8}x + \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) \right]_0^{\pi/4} = \frac{3\pi}{32} + \frac{1}{4}$
- e) $\int \frac{1}{x^2 + 4} dx = \int \frac{1}{4 \left(\frac{x}{2}\right)^2 + 1} dx = \frac{1}{4} \cdot 2 \cdot \operatorname{arctg}(x/2) + C = \frac{1}{2} \operatorname{arctg}(x/2) + C$
- f) $\int \frac{1}{\sqrt{1+3x^2}} dx = \int \frac{1}{\sqrt{1+(\sqrt{3}x)^2}} dx = \frac{1}{\sqrt{3}} \operatorname{arsh}(\sqrt{3}x) + C$

$$g) \int_{-1}^0 (2x+1)^{10} dx = \left[\frac{1}{2} \cdot \frac{1}{11} (2x+1)^{11} \right]_{-1}^0 dx = \frac{1}{11}$$

$$h) \int \frac{x^3-1}{x+2} dx = \int x^2 - 2x + 4 - \frac{9}{x+2} dx = \frac{1}{3}x^3 - x^2 + 4x - 9 \ln|x+2| + C$$

$$4. a) \int_1^\infty \frac{1}{x^2+1} dx = \lim_{b \rightarrow \infty} \left[\operatorname{arctg} x \right]_1^b = \lim_{b \rightarrow \infty} \operatorname{arctg} b - \operatorname{arctg} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$b) \int_0^1 \frac{1}{\sqrt[3]{x^2}} dx = \lim_{b \rightarrow 0^+} \int_b^1 x^{-2/3} dx = \lim_{b \rightarrow 0^+} \left[3x^{1/3} \right]_b^1 = \lim_{b \rightarrow 0^+} 3 - 3\sqrt[3]{b} = 3$$

$$c) \int_e^\infty \frac{1}{x \ln x} dx = \lim_{b \rightarrow \infty} \int_e^b \frac{1}{\ln x} \cdot \frac{1}{x} dx = \lim_{b \rightarrow \infty} \left[\ln \ln x \right]_e^b = \lim_{b \rightarrow \infty} \ln \ln b - \ln \ln e = \infty$$

$$d) \int_e^\infty \frac{1}{x(\ln x)^2} dx = \lim_{b \rightarrow \infty} \int_e^b (\ln x)^{-2} \frac{1}{x} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{\ln x} \right]_e^b = \lim_{b \rightarrow \infty} -\frac{1}{\ln b} + \frac{1}{\ln e} = 1$$

$$5. a) f(x) = \int 4x + \sin(2x) dx = 2x^2 - \frac{1}{2} \cos(2x) + C, \text{ és } 0 = f(0) = -\frac{1}{2} + C, \text{ tehát } C = \frac{1}{2}, \\ \text{és } f(x) = 2x^2 - \frac{1}{2} \cos(2x) + \frac{1}{2}.$$

$$b) f'(x) = \int 6x^2 + x^{-3/2} dx = 2x^3 - 2x^{-1/2} + C, \text{ és } 2 = f'(1) = 2 - 2 + C, \text{ így } C = 2, \text{ és} \\ f'(x) = 2x^3 - 2x^{-1/2} + 2. \text{ Ebből } f(x) = \int 2x^3 - 2x^{-1/2} + 2 dx = \frac{1}{2}x^4 - 4x^{1/2} + 2x + D, \\ \text{és } 0 = f(1) = \frac{1}{2} - 4 + 2 + D, \text{ tehát } D = \frac{3}{2}, \text{ és } f(x) = \frac{1}{2}x^4 - 4\sqrt{x} + 2x + \frac{3}{2}.$$