Rough Syllabus, Part 1

Linear spaces

1. Definition. Let H be a non-empty set and K a number field. (In this semester we deal only with the cases when the field is $\mathbf{R}$, the real number field, or $\mathbf{C}$, the complex number field.) The set H is a **linear space** (or **vector space**) over K, if there is defined a commutative, associative, invertable operation on H which is called **addition**, i.e. for any $a, b, \in H$ their sum $a + b \in H$, and

i) $a + b = b + a$;

ii) $(a + b) + c = a + (b + c)$;

iii) There is an element $o \in H$ such that $a + o = a$ for every $a \in H$;

iv) For every $a \in H$ there is an $a^- \in H$ such that $a + a^- = o$.

There is also defined the **constant multiple** of the elements of H, i.e. for any $a \in H$ and $\alpha \in K$ the "product" $\alpha a \in H$, and

i) $(\alpha + \beta)a = \alpha a + \beta a$;

ii) $(\alpha\beta)a = \alpha(\beta a)$;

iii) $\alpha(a + b) = \alpha a + \beta a$;

iv) $1a = a$.

Corollaries:

1. There is only one o in H.
2. Every a has only one inverse.
3. If $a + b = a$, then $b = o$.
4. $0a = o$.
5. $a^- = (-1)a = -a$.

Examples for linear spaces: $n - tuples$ over the field from which they have their elements; real polynomials of degree not more than 3; real or complex polynomials; continuous real functions on an interval $[a, b]$.

2. Definition. A subset A of H is called a **subspace** of H, if it itself is a vector space with respect to the operations defined in H.

3. Definition. The o vector (defined in iii) for addition) is called **null vector** or **zero vector**.

Remark: The o vector itself form a linear space.

4. Definition. The two subsets of H, the o vector and H itself are called the **trivial subspaces** of H.

Remark: If A is a subset of H, then in order to decide whether it is a subspace or not, it is enough to check whether it is closed for addition and number multiplication (since the properties of sum and constant multiple are satisfied already for the total H).

1. Theorem. If V_1 and V_2 are subspaces of a linear space, then $V_1 \cap V_2$ is also a subspace.

5. Definition. If two subspaces V_1 and V_2 have the property, that $V_1 \cap V_2 = \mathbf{o}$, and every vector $\mathbf{w}$ in the space can be given in the form $\mathbf{w} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$ where $\mathbf{v}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$, then V_1 and V_2 are called **complementary subspaces** of each other.

Examples: $n - tuples$ having 0 on the first position form a subspace in the space of $n - tuples$; among the real polynomials of degree not more than 3 the polynomials having root $x = 1$ form a subspace; among real or complex polynomials the ones having 0 constant term form a subspace; among continuous real functions on an interval $[a, b]$ the functions of form $\alpha \cos x + \beta \sin x$ form a subspace; polynomials having roots 1 and polynomials having root 2 form complementary subspaces in the space of real polynomials of degree not more than 1.

6. Definition. Vector b is a **linear combination** of vectors a_1, a_2,...a_n if $b = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n$.

7. Definition. ($\mathbf{D}_1$) Vectors a_1, a_2,...a_n are **linearly independent**, if from the equality $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = o$ it follows that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

8. Definition. ($\mathbf{D}_2$) Vectors a_1, a_2,...a_n are **linearly independent** if
in the case $n = 1$ $a_1 \neq o$;
in the case $n \geq 2$ none of them can be given as a linear combination of the others.

2. Theorem. Definitions $\mathbf{D}_1$ és $\mathbf{D}_2$ are equivalent.

I.e. any of them is chosen as definition, the other one becomes a corollary, i.e. it becomes a theorem.

Remark: A system of linearly independent vectors is called a **linearly independent system**.
Linearly independent system can not contain the null vector.
In the special case, when the system consists of two vectors, they are independent if none of them is a constant multiple of the other.

3. Theorem. If vector b is a linear combination of the linearly independent system a_1, a_2,...,a_n, then this representation is unique.

It means, the constant multipliers are unique. Consequently, if the constant multipliers are different in two linear combinations of the linearly independent vectors a_1, a_2,...a_n, then the results are two different vectors.

9. Definition. The **rank** of the system of vectors b_1, b_2,...,b_r is n, if the maximal number of linearly independent elements in it is n.

4. Theorem. Such a maximal linearly independent system can be chosen usually in several different ways, but it has always the same number of elements.

5. Theorem. -If some elements of a vector system are left out, the rank of the system can not increase.
-If some vectors are joined to a system of vectors, the rank of the system can not decrease.
-If any element of a system is multiplied by a nonzero number, the rank of the system does not change.
-If any element of the system is added to others, the rank of the system does not change.
-If such elements are left out, which are linear combinations of the remaining ones, the rank of the system does not change.
-If such elements are joined to the system, which are linear combinations of the original ones, the rank of the system does not change.

6. Theorem. The all possible linear combinations of a system of vectors a_1, a_2,...a_n form a subspace in the space.

10. Definition. If V is the subspace of all possible linear combination of the vectors a_1, a_2,...a_n, then these vectors are called a **generator system** of the subspace, and the subspace is **spanned** or **generated** by this system of vectors.

11. Definition. If every elements of the linear space H can be given as a linear combination of the linearly independent vectors a_1, a_2,...,a_n, then the system a_1, a_2,...a_n is called a **basis** (or **fundamental system**) of H.

Corollary: A basis is a generator system of the space consisting of linearly independent elements.

7. Theorem. If vector space H has a basis (fundamental system) with n elements, then every basis of H consists of n elements.

12. Definition. The **dimension** of H is n, if H has a basis of n elements, i.e. if the minimal number of vectors in a generator system of it has n elements.
It is obviously the same as the maximal number of vectors in a linearly independent system in this space.

Remark: There are vector spaces with dimension infinity. Now, we deal only with finite dimensional spaces, infinite dimensional spaces are only mentioned.

Remark: The space generated by the empty set is the null vector. The space consisting only of the null vector has dimension zero. This space does not have any linearly independent system (not even one element).

Examples: The dimension of $n-tuples$ over the field from which they have their elements is n; The dimension of real polynomials of degree not more than 3 is 4; The dimension of real polynomials is infinite; The dimension of continuous real functions on interval $[a, b]$ is infinite; The dimension of complex $n-tuples$ over the real field is $2n$; The dimension of real functions having the form $\alpha \cos x + \beta \sin x$ is two.

13. Definition. If a basis (fundamental system) is fixed in H and the order of the elements is fixed in this system, then every vector in H can be uniquely defined by the constants occuring in its representation with this system. These constants are called the **coordinates** of the given vector with respect to the given basis (fundamental system).

Calculations with vectors given by coordinates: The coordinates of a sum vector are the sum of the coordinates of the addend, the coordinates of a constant multiple of a vector are the constant multiples of the coordinates of this vector.

When the basis is changed, the coordinates of the vectors change.