Rough Syllabus, Part 5
Inner product (or scalar product) in linear spaces

The simple definition of inner product (scalar product or dot product), what we have learned last semester for the vectors represented with oriented line segments is the Euclidean space around us, obviously does not work for general linear spaces. We can define an inner product in a general linear space, if we keep the properties of the dot product.

1. Definition. Inner product in linear spaces over the complex numbers:
Let V be a linear space over complex numbers. A two variable complex valued function $\langle \mathbf{a}, \mathbf{b} \rangle$ ($\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$) is an inner product in V, if the following properties are valid:
i) $\langle \mathbf{a}, \mathbf{b} \rangle = \overline{\langle \mathbf{b}, \mathbf{a} \rangle}$
ii) $\langle \mathbf{a}, \mathbf{a} \rangle \geq 0$,
iii) $\alpha \langle \mathbf{a}, \mathbf{b} \rangle = \langle \alpha \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{a}, \overline{\alpha}\ \mathbf{b} \rangle$
iv) $\langle \mathbf{a}, \mathbf{b} + \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{a}, \mathbf{c} \rangle$

Since we discuss only a very simple and short theory from this topic, for us it will be more simple in notations to apply the property iii) in the form: iii*) $\alpha \langle \mathbf{a}, \mathbf{b} \rangle = \langle \overline{\alpha}\ \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{a}, \alpha \mathbf{b} \rangle$

For real numbers (the conjugate is the number itself):

2. Definition. Inner product in linear spaces over the real numbers:
Let V be a linear space over real numbers. A two variable real valued function $\langle \mathbf{a}, \mathbf{b} \rangle$ ($\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$) is an inner product in V, if the following properties are valid:
i) $\langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{a} \rangle$
ii) $\langle \mathbf{a}, \mathbf{a} \rangle \geq 0$,
iii) $\langle \alpha \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{a}, \alpha \mathbf{b} \rangle = \alpha \langle \mathbf{a}, \mathbf{b} \rangle$
iv) $\langle \mathbf{a}, \mathbf{b} + \mathbf{c} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{a}, \mathbf{c} \rangle$

Consequence: $\langle \mathbf{a}, \mathbf{a} \rangle = 0$ if and only if $\mathbf{a} = \mathbf{o}$.

1. Theorem. Cauchy-Schwarz inequality
For reals:
$\langle \mathbf{a}, \mathbf{a} \rangle \langle \mathbf{b}, \mathbf{b} \rangle \geq (\langle \mathbf{a}, \mathbf{b} \rangle)^2$ For complex:
$\langle \mathbf{a}, \mathbf{a} \rangle \langle \mathbf{b}, \mathbf{b} \rangle \geq \langle \mathbf{a}, \mathbf{b} \rangle \overline{\langle \mathbf{a}, \mathbf{b} \rangle}$

There is not too much reason to talk about the size of angles in higher dimension, or e.g. between polynomials or continuous function above an interval, but :

3. Definition. Two elements $\mathbf{a}$ and $\mathbf{b}$ of a linear space with an inner product are orthogonal or perpendicular to each other, if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$.

The most often applied inner product among real $n - tuples$ is $\langle \mathbf{a}, \mathbf{a} \rangle = \sum_{i=1}^{n} a_i b_i$.

The most often applied inner product among complex $n-tuples$ applying iii*) $\langle \mathbf{a}, \mathbf{a} \rangle = \sum_{i=1}^{n} \overline{a_i} b_i$

If vectors are given in matrix form by coordinates as columns, then $\langle \mathbf{a}, \mathbf{a} \rangle = \mathbf{a}^T \mathbf{b}$ in the real case, and $\langle \mathbf{a}, \mathbf{a} \rangle = \mathbf{a}^* \mathbf{b}$ in the complex case.

We know, that when we change the basis, the coordinates change. E.g. Consider an n-dimensional real space. The coordinates of the first and second basis vectors are always

$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$. Applying the inner product given above as for $n - tuples$ they are orthogonal.

The first two vectors in the new basis will have these coordinates in the new basis. If they are not orthogonal, their inner product can not be calculated in the way as it given above. How we should calculate the inner product with coordinates with respect to the new basis, if we want to keep the inner product given in the old basis? Remember, that the formula to get the coordintes in the new basis from the old coordinates is: $\mathbf{x}_{new} = B^{-1}\mathbf{x}_{old}$.
Then

$\langle \mathbf{x}_{new}, \mathbf{y}_{new} \rangle = \mathbf{x}^*_{new}\mathbf{y}_{new} = \mathbf{x}^*_{new} I^* I \mathbf{y}_{new} = \mathbf{x}^*_{old}(BB^{-1})^*(BB^{-1})\mathbf{y}_{old} =$

$\mathbf{x}^*_{old}(B^{-1})^*(B^*B)B^{-1}\mathbf{y}_{old} = \mathbf{x}^*_{new}(B^*B)\mathbf{y}_{new}$

The matrix on the middle is called **weight matrix**. In the matrix $W = B^*B$ the element $w_{i,j}$ is the inner product of the i-th and the j-th basis vectors of the old basis, wich values are known from the original basis.

Corollary: An inner product can be defined by any invertible matrix in the way given above. I.e. infinitely many different inner product can be defined in a space.

In the linear space of functions continuous on the interval $[a, b]$ the most often used inner product is $\langle f, g \rangle = \int_a^b f(x)p(x)g(x)dx$, where $p(x) > 0$ continuous. This p is called **weight function**. Also for functions infinitely many different inner product can be defined.

Now, we can consider the system of linear equations in an other way. Every equation of a homogeneous sytem can be cosidered as the inner product of the corresponding row vector of the coefficient matrix with the unknown vector.
In the homogeneous case we are looking for a vector orthogonal to every row of the coefficient matrix. So we are looking for the vectors of such complementary subspace V_c of the subspace generated by the rows V_r, which has perpendicular vectors to each vector in V_r. Now it is more obvious: The more is the dimension of the subspace of the rows, the less is the dimension of the space being orthogonal to it, since the sum of the dimensions is n. If the rank of the coefficient matrix is n, then only the zero vector can be perpendicular to a basis of the space.
In the inhomogeneous system the inner products with the row vectors are given. It practically means that the perpendicular projections of the unknown vector given to the rows are given. If there are given projections into n linearly different directions, then the unknown vector is uniquely defined. Warning: These projections are not perpendicular to each other, so the solution is not a simple sum of them. In this case, if the rows are not linearly independent, then the informations about the projections can contradict to each other, then the system does not have any solution.

If an inner product is defined in a space, then for any system of vectors there is a system of orthogonal vectors such that they generate the same subspace. A wellknown methos is the Gram-Schmidt process.