Introduction to Algebra **Example sheet 9** **Autumn 2025**

1. Compute the minimal and characteristic polynomials of the following matrices and determine their eigenvalues. (a, b are real parameters)

$$A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix};\ B=\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix};\ C=\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix};\ D=\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix};\ E=\begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

2. True or False?

(i) The minimal polynomial is always irreducible.

(ii) If $f(t) \in F[t]$ is irreducible and $f(\varphi) = 0$ then $f(t) = m_\varphi(t)$.

(iii) If $\chi_\varphi(t)$ has no multiple root then $m_\varphi(t) = \chi_\varphi(t)$.

(iv) If $f(t) \in F[t]$ is such that $f(A) = 0$ then $f(\lambda) = 0$ for every eigenvalue λ of A.

3. How can one see from the minimal polynomial of A if A is invertible? Show that if A is invertible then there is a polynomial $f(t) \in F[t]$ such that $A^{-1} = f(A)$.

4. Determine the 2×2 rational matrices A for which $A^5 = I$.

5. A linear transformation $\varphi \in \mathrm{Hom}(V)$ is a projection if $\varphi = \varphi \circ \varphi$ (aslo called φ^2).

(i) Show that φ is a projection if and only if $\mathrm{id}_V - \varphi$ is a projection.

(ii) Show that φ is a projection if and only if $\varphi|_{\mathrm{Im}(\varphi)} = \mathrm{id}_{\mathrm{Im}(\varphi)}$.

(iii) Show that if φ is a projection then $V = \mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\varphi)$.

(iv) What is the minimal polynomial of a projection?

6. HW Compute the minimal and characteristic polynomials of the following matrices and determine their eigenvalues.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix};\ B = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix};\ C = vw^T,\ (v, w \in \mathbb{R}^n).$$

7. Suppose $A \in M^{n\times n}(F)$ and $\deg(m_A(t)) = k$. What are the possible values for $\deg(m_{A^2}(t))$?

8. Let $f(t) \in F[t]$, $n = \deg(f) \geq 1$. Show that there exists an $n \times n$ matrix A with $m_A(t) = f(t)$.