Introduction to Algebra **Example sheet 8** **Autumn 2025**

1. Which of the following are linear maps from the vector space $M = F^{3\times 3}$ to itself? If it is then determine the kernel and image.

(i) Fix $B \in M$ and $\varphi(A) = AB$;

(ii) Fix $B \in M$ and $\psi(A) = BA$;

(iii) $\eta(A) = A^2$;

(iv) $\xi(A) = A^T$;

(v) $\nu(A)$ is A, but all its last column is replaced by 0's.

2. True or False? ($\varphi \in \mathrm{Hom}(V, W)$ and $v_1 \dots v_k \in V$.)

(i) If $v_1, \dots, v_k$ are independent in V then $\varphi(v_1), \dots, \varphi(v_k)$ are independent in W.

(ii) If $\varphi(v_1), \dots, \varphi(v_k)$ are independent in W then $v_1, \dots, v_k$ are independent in V.

(iii) If $v_1, \dots, v_k$ span V then $\varphi(v_1), \dots, \varphi(v_k)$ span W.

(iv) If $v_1, \dots, v_k$ span V then $\varphi(v_1), \dots, \varphi(v_k)$ span $\mathrm{Im}(\varphi)$.

(v) If $\varphi(v_1), \dots, \varphi(v_k)$ span $\mathrm{Im}(\varphi)$ then $v_1, \dots, v_k$ span V.

3. Suppose $\dim(V) = n$ and $\dim(W) = k$. What is the dimension of $\mathrm{Hom}(V, W)$?

4. Suppose $U, W \le V$.

(i) Show that $\dim(U) + \dim(W) \ge \dim(U + W)$.

(ii) Show that if $V = U \oplus W$ then $\dim(U) + \dim(W) = \dim(V)$.

(iii) In fact, show that $\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W)$ always holds.

5. Suppose $\varphi \in \mathrm{Hom}(V, V)$ (usually we call it $\mathrm{Hom}(V)$ for brevity). Then $\mathrm{Ker}(\varphi), \mathrm{Im}(\varphi) \le V$. Give and example when $V = \mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\varphi)$ and give an example when the sum is not direct.

6. HW Let V be the $\mathbb{F}_2$-vectorspace consisting of subsets of $\{a; b; c; d; e; f\}$ as in the Study Room sheet 8 exercise 1.(xii). Define an isomorphism between V and $\mathbb{F}_2^6$. Verify its properties.

7. Let $\varphi \in \mathrm{Hom}(V, W)$. Suppose $v_1, \dots, v_k$ are independent in V and $\varphi(v_i) = \varphi(v_j)$ for all $1 \le i, j \le k$. Show that $\dim \mathrm{Ker}(\varphi) \ge k - 1$.

8. Suppose V, W are vector spaces over the same field F. Show that one of them is isomporhic to a subspace of the other.