Introduction to Algebra **Example sheet 7** **Autumn 2025**

1. Which of the following are subspaces of the vector space $M = F^{100\times 100}$? And if not, why?

(i) $\{A \in M \mid AB = BA\}$, when B is a fixed matrix from M;

(ii) $\{A \in M \mid AB = 0\}$, when B is a fixed matrix from M;

(iii) $\{A \in M \mid A^2 = 0\}$;

(iv) $\{A \in M \mid \mathrm{rk}(A) \leq 3\}$;

(v) diagonal matrices;

(vi) diagonalisable matrices;

(vii) lower triangular matrices;

(viii) strictly lower triangular matrices;

(ix) skew-symmetric matrices.

2. Find two nontrivial subspaces of $\mathbb{F}_3^2$ which are direct complements. (That is, their direct sum is the whole space $\mathbb{F}_3^2$.)

3. Let V be the vector space of rational polynomials (over $\mathbb{Q}$) of degree at most 3. Let W denote the subspace of those polynomials divisible by $x+1$, U the subspace of polynomials divisible by x^2+1 and Y the subspace of polynomials divisible by x^2-1. Determine $W+U$, $W+Y$ and $U+Y$. Which sum is direct?

4. Recall from last semester how to do this!

Choose a maximal linearly independent system of the columns of the following matrix A! Write the other columns as linear combinations of the previous ones. Compute a basis of $\mathcal{N}(A) = \mathrm{Ker}(A)$!

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 2 & -1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

5. HW Enumerate all the subspaces of $\mathbb{F}_2^3$ and find among them pairs of direct complements.

6. Verify that
a) the intersection of two affine subspaces is an affine subspace or empty, but
b) the union of two subspaces is a subspace if and only if one contains the other
c) (if the underlying field has more than 2 elements then) the union of two affine subspaces is an affine subspace if and only if one contains the other.

7. Is there a polynomial f of degree 100 such that its remainder after division by $x^{60}-2$ and by $x^{49}+3$ are the same?