Introduction to Algebra **Example sheet 6** **Autumn 2025**

1. Suppose $f(x) \in \mathbb{R}[x]$ of degree n has (complex) roots $\alpha_1, \dots, \alpha_n$. Show that $\alpha_1^k + \alpha_2^k + \cdots \alpha_n^k$ is real for all k.

2. Suppose the monic $f(x) \in \mathbb{C}[x]$ of degree n has roots $\alpha_1, \dots, \alpha_n$ such that $\alpha_1^k + \alpha_2^k + \cdots \alpha_n^k$ is real for all k. Show that $f(x) \in \mathbb{R}[x]$.

3. Factorise $x^4 + 1$ into a product of irreducibles over $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively.

4. Show that the roof α of $f(x)$ has multiplicity k if and only if $x - \alpha | f^{(j)}(x)$ for $j = 1, \dots, k-1$, but $x - \alpha \nmid f^{(k)}(x)$.

5. Let α, β, γ be the roots of $x^3 + ax^2 + bx + c$. Determine the monic polynomial of degree 3 whose roots are

$$\alpha + \beta, \quad \beta + \gamma, \quad \alpha + \gamma.$$

6. Using the Schönemann-Eisenstein criterion for suitable primes or otherwise show that $\Phi_n(x)$ is irreducible over $\mathbb{Z}$ for $n \leq 12$.

7. HW Let α, β, γ be the roots of $x^3 + ax^2 + bx + c$. Determine the monic polynomial of degree 3 whose roots are

$$\frac{\alpha + \beta}{\gamma}, \quad \frac{\beta + \gamma}{\alpha}, \quad \frac{\alpha + \gamma}{\beta}.$$

8. Prove the "other" Schönemann-Eisenstein criterion: Let p be a prime and $f(x) \in \mathbb{Z}[x]$, $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ such that p does not divide a_0, but divides a_i for $i > 0$ and p^2 does not divide a_n. Prove that if $f(x) = g(x)h(x)$ then $g(x)$ or $h(x)$ is of degree 0.

9. Let $x_1, \dots x_n$ be indeterminates and e_k denote the k-th elementary symmetric function, p_k the k-th power sum and h_k the sum of all monomials of dgree k (these are called the *complete* symmetric functions). For a new variable t, define

$$E(t) = 1 + \sum_{k=1}^{n} e_k t^k, \quad P(t) = \sum_{k=1}^{\infty} \frac{p_k}{k} t^k, \quad H(t) = 1 + \sum_{k=1}^{\infty} h_k t^k.$$

The first is a polynomial, the other two are power series! Show that

$$E(t) = \prod_{i=1}^{n} (1 + x_i t), \; H(t)E(-t) = 1 \text{ and } P(t) = \log H(t).$$

10. Let m, n be positive integers. Suppose that the complex numbers $z_1, \dots, z_n$ are such that

$$z_1^k + z_2^k + \cdots + z_n^k = m, \quad (\forall 1 \leq k \leq n).$$

Prove that $z_1, \dots, z_n$ are the roots of the polynomial

$$f(x) = x^n - \binom{m}{1} x^{n-1} + \cdots + (-1)^k \binom{m}{k} x^{n-k} + \cdots + (-1)^n \binom{m}{n}.$$