Introduction to Algebra **Example sheet 5** **Autumn 2025**

1. Determine $\Phi_n(x)$ for $n = 8, 9, 10, 12$.

2. Determine $\Phi_p(x)$ for p a prime.

3. Determine the monic polynomials of lowest degree a) in $\mathbb{C}[x]$ and b) in $\mathbb{R}[x]$, for which i is a double root and 1 is a triple root!

4. How many irreducible factors does the polynomial $-6x^3 + 6x^2 - 12$ have in $\mathbb{Q}[x]$, $\mathbb{Z}[x]$, $\mathbb{R}[x]$ and $\mathbb{C}[x]$?

5. What is $\gcd(-6x^3 + 6x^2 - 12, 3x^2 - 3x - 6)$ in $\mathbb{Q}[x]$ and $\mathbb{Z}[x]$?

6. Let n be an integer, $\varepsilon = e^{\frac{2\pi i}{n}}$.

(i) Confirm that the n-th roots of 1 are the powers of ε.

(ii) Show that $o(\varepsilon^k) = \frac{n}{(n,k)}$.

(iii) In particular, derive that $\deg \Phi_n(x) = \varphi(n)$.

7. Show that $\Phi_n(x)$ is irreducible in $\mathbb{Q}[x]$ if $n = 1, 2, 3, 4, 6$. Show also for $n = 5, 8, 10$.

8. **HW** Let n be odd. What is the relation between $\Phi_n(x)$ and $\Phi_{2n}(x)$?

9. Prove that $\Phi_{12}(x) \pmod p$ is reducible in $\mathbb{F}_p[x]$ for every prime p.

10. Finish the proof from the lecture. Prove that if $f(x) \in \mathbb{Z}[x]$ has two factorisations into irreducible integer polynomials then these possibly differ only in order and sign.