Introduction to Algebra **Example sheet 3** **Autumn 2025**

1. Show that if a and b are coprime (i.e. $(a,b)=1$) and ab is a square then a and b are both squares. Show that if x is an even square then $x-1$ is not a cube (except, of course, for $-1=(-1)^3$).

2. What is the time at $39^{38^{37}}$ minutes after midnight? (Hint: There are 1440 minutes in a day.)

3. HW What are the last three digits of 1234^{9876}?

4. Let p be a prime and let $\mathcal{H}=\{0,1,\dots,p-1\}$. Let addition, subtraction and multiplication be defined on $\mathcal{H}$ naturally: $a+b=c$ if $a+b\equiv c \pmod p$, etc. Argue that division (by non-0) can also be defined and that makes $\mathcal{H}$ a field.

5. Suppose $m>3$ is composite and $\{a_1,\dots a_{\varphi(m)}\}$ a reduced residue system modulo m. Determine the residue of their product $\prod a_i$ modulo m.

6. Show that the system of two congruences

$$x\equiv a_1 \pmod{m_1}, \qquad x\equiv a_2 \pmod{m_2}$$

is solvable if and only if $(m_1,m_2)\mid a_1-a_2$. If this holds then the solution is unique modulo $[m_1,m_2]$.

7. What can be the number $\overline{ab}_7=\overline{ba}_{10}$?

8. Is there a limit on the length of a number that is in base 10 and equal to some rearrangement of its digits in base 7? (In the previous exercise we had length 2.)

9. Let $m>2$ and $\{a_1,\dots a_{\varphi(m)}\}$, $\{b_1,\dots b_{\varphi(m)}\}$ be two reduced residue systems modulo m.

(i) Show that if m is a prime then there exist two indices $i\neq j$ such that $a_ib_i\equiv a_jb_j \pmod m$.

(ii) Show that the above holds for every $m>2$.

10. Show that for positive odd integers a,b we have $(2^a-1,2^b+1)=1$.

11. Show that for positive integers a,b we have $(3^a-1,3^b-1)=3^{(a,b)}-1$.

12. Show that $641|2^{32}+1$ using that $641=5^4+2^4=5\cdot 2^7+1$. (Euler)

13. Show that if a and b are positive integers such that $\varphi(a)=\varphi(b)=n$ then there exist n integers that form a r.r.s. both modulo a and modulo b