1. Solve the congruences (in $\mathbb{Z}$)!

(i) $12x \equiv 15 \pmod{21}$,

(ii) $12x \equiv 4 \pmod{6}$,

(iii) $12x \equiv 4 \pmod{2}$ and

(iv) $30x \equiv 4 \pmod{37}$.

2. Compute the greatest common divisor of the following pairs of integers or expressions. (Try to do this both with the smallest non-negative remainder and with the remainder of smallest absolute value.) Express it as an integral combination of the original numbers (i.e. as $(a, b) = xa + yb$).

(i) 2020, 1974;

(ii) 89, 55;

(iii) n, $n + 1$;

(iv) $2n$, $3n + 1$;

3. At most how many steps does the Euclidean Algorithm need to reach the greatest common divisor of two numbers less then 100?

4. HW

(i) What is the exponent of 2 in the canonical representation of 17! (17 factorial)?

(ii) How many zero digits are at the end of the number 100! (100 factorial)?

(iii) What is 2^{67} modulo 61?

5. We know that for three integers a, b, c the linear equation $ax + by = c$ is soluble among the integers if and only if $(a, b)|c$. Show that if $a, b > 0$ and $c > ab$ then the equation is soluble among the *positive* integers.
What is the smallest c_0 such that for every $c \geq c_0$ the equation $ax + by = c$ is soluble among the non-negative integers?

6. Prove that if 23 divides $5a + 9b$ for some integers a, b, then 23 also divides $3a + 10b$!

7. Suppose that $(a, b) = 5$. What can $(a + b, a - b)$ be?

8. Show that for all $n \in \mathbb{N}$ we have $133|11^{n+1} + 12^{2n-1}$!

9. Prove the following claim using induction. The sum of the first few cubic numbers is always a square.

10. Determine all solutions of the following equality, where a, b are positive integers and p is a positive prime.

$$[a, b] + (a, b) = a + b + p$$

11. Prove that $\varphi(ab) = \varphi(a)\varphi(b)$ if a and b are coprime.