Introduction to Algebra **Example sheet 1** **Spring 2025**

1. True or false?

(i) If $(a, b) = c$ then $(\frac{a}{c}, \frac{b}{c}) = 1$.

(ii) If $(a, b) = d$ then $(\frac{a}{d}, b) = 1$ or $(a, \frac{b}{d}) = 1$.

(iii) $e|ab$ if and only if $\frac{e}{(e,a)}|b$.

(iv) $f|ab$, $(a, b) = 1$ implies that $f|a$ or $f|b$.

2. Compute the gcd of 133 and 73. Can you find two numbers below 100 for which the Euclidean Alorithm takes six steps? Seven? What is the maximum?

3. Show that a positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9. Further, show that the remainder of an integer modulo 9 is the same as the remainder of the sum of its digits modulo 9. (Example: $127 = 14 \cdot 9 + 1$ and $1 + 2 + 7 = 10 = 1 \cdot 9 + 1$.)

4. Show that any 6-digit number we obtain by repeating the digits of a 3-digit number is divisible by 91. For example by repeating 123 we get $123123 = 1353 \cdot 91$.

5. What is the gcd and the lcm of 3718 and 3234?

6. HW Show that the quotient $\frac{3n+5}{7n+12}$ can never be reduced.

7. John is standing at a tap of water with two large bottles, one of 4 litres and one of 3. He needs exactly 2 litres in one of them. How can he manage? What if his bottles are of 12 and 15 litres? How can we generalise?

8. Bilbo picked his favourite positive integer up to 111. Frodo can ask about it only yes-no questions. How many questions are needed?

9. Having seen the success of Frodo, Bilbo became uneasy so now he demands that Frodo should disclose the questions before he picks his number. Now many predetermined questions of Frodo are needed so that he can guess every positive integer up to 111?

10. Prove that $\sqrt{2}$ is irrational.

11. Let $f_0 = 0$, $f_1 = 1$ and for every $n > 1$ let $f_n = f_{n-1} + f_{n-2}$; this is the Fibonacci sequence. Verify that for every integer k there are infinitely many indices n such that $k|f_n$.

12. Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a*b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$. (Putnam)