1. True or false for complex sesquilinear functions?

(i) If B is diagonal in a basis then B is Hermitian.

(ii) If B is Hermitian then its matrix has positive real determinant.

(iii) If $B(v,w) = 0$ for some $v, w \in \mathbb{C}^n$ then $B(w,v) = 0$.

(iv) If B is Hermitian and $B(v,w) = 0$ for some $v, w \in \mathbb{C}^n$ then $B(w,v) = 0$.

(v)* If $B(v,w) = 0$ implies $B(w,v) = 0$ for $v, w \in \mathbb{C}^n$ then cB is Hermitian for some $0 \neq c \in \mathbb{C}$.

2. Verify the following for subspaces U_1, U_2 of a (real/complex) Euclidean space V.

(i) $U_1 \leq U_2 \Leftrightarrow U_1^\perp \geq U_2^\perp$.

(ii) $(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$.

(iii) $(U_1 \cap U_2)^\perp = (U_1^\perp + U_2^\perp)$.

3. Verify the following two claims for vectors of a complex Euclidean space V.

(i) $iv + w \perp v - iw$ if and only if $v = iw$.

(ii) $v + iw \perp v - iw$ if and only if $||v|| = ||w||$ and (v, w) is purely imaginary..

4. Convert the following real quadratic functions to a sum of signed squares using (orthogonal) diagonalisation.
(i) x_1x_2; (ii) $2x_1x_2 - 2x_2x_3$; (iii) $x_1^2 - x_3^2 + 2x_1x_2 + 2x_1x_3$;
(iv) $x_1^2 + x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$.

5. HW Convert the real quadratic function $f(x, y, z, u) = xy + yz + zu + ux$ to a sum of signed squares using (orthogonal) diagonalisation. Detail your computation.

6. Let $V = \mathbb{C}^n$ a complex Euclidean space. Show that the following are equivalent for a $\varphi \in \mathrm{Hom}(V)$.

(i) φ is normal;

(ii) $||\varphi(v)|| = ||v||$ for all $v \in \mathbb{C}^n$.

(iii) There exists $f(x) \in \mathbb{C}[x]$ such that $\varphi^* = f(\varphi)$.

7. Suppose $A, B \in M^{n\times n}(\mathbb{C})$ are normal and $AB = BA$. Show that AB and $A + B$ are also normal and there exists an orthonormal basis where both A and B become diagonal (a common eigenbasis).

8. What is the set of values $\{Q(v) \mid v \in \mathbb{C}^n\}$ of a Hermitian quadratic form Q? Give an example of a complex sesquilinear function B such that the set of values of its quadratic form is $\{Q(v) \mid v \in \mathbb{C}^n\} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) = \mathrm{Im}(z)\}$, a line on the complex plane.