Introduction to Algebra 2. **Midterm 2** **30.05.2023**

Only empty sheets of size A4 and pens can be used. No other exam aids (e. g. calculators, notes) are permitted! Communication with others is also forbidden.

There are 6 problems (**not** ordered by difficulty), each is worth 1 point. To pass, you have to obtain at least 1 point. Test time is 90 minutes.

1. Which of the following matrices is/are unitarily diagonalizable? Compute the diagonal matrix (but not the decomposition U^*DU) if it is possible!

$$\begin{pmatrix} 1 & i & 1 \\ 0 & 1 & i \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & i & 1 \\ -i & 1 & -i \\ 1 & i & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & i & 1 \\ i & 1 & -i \\ 1 & -i & 1 \end{pmatrix}$$

2. Diagonalize the quadratic form $q(x_1, x_2, x_3) = x_1^2 + 2x_1x_2 + 4x_1x_3 + 2x_2^2 + 2x_2x_3 + x_3^2$! Compute a basis in which it is diagonal. What is its signature?

3. Consider the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a^2 \end{pmatrix}$. Compute an LU decomposition for $a = 0$. For which values of a does it have a Cholesky decomposition (LL^*)?

4. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$. Which of the following is the pseudoinverse of A? Why?

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \frac{1}{6}\begin{pmatrix} 8 & 2 & -4 \\ -3 & 0 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1/2 & 1/3 \end{pmatrix}$$

   Find the closest solution of the system of linear equations given by $Ax = e_1$!

5. Compute the singular values and a pair of singular vectors of the matrix $\begin{pmatrix} 0 & 0 & 3 \\ 2 & 2 & 1 \end{pmatrix}$

6. Compute the polar decomposition ($A = UP$) of the matrix $A = \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$!

Jó munkát!