Intro to Algebra 2 **Topics** **Spring 2024**

1. Determinant of a product, Vandermonde determinant. Block matrices. Existence of basis and dimension. (Using Zorn's Lemma.)

2. Matrix of a linear map/transformation. Multiplication of matrices and maps. Dimension of $\mathrm{Hom}(V_1, V_2)$, $\mathrm{Hom}(V)$. Change of basis, similarity of matrices in $M_{n,k}(F)$ and $M_n(F)$. Invariance of trace, determinant, rank and nullity. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

3. Eigenvector, eigenvalue, eigenspace. Eigenbasis, diagonalisability. $f(\varphi)$ and $f(A)$ for $f \in F[x]$, $\varphi \in \mathrm{Hom}(V)$ and $A \in M_n(F)$. Minimal polynomial and its roots.

4. Characteristic polynomial of A and of φ and its roots. Cayley-Hamilton Theorem. Algebraic and geometric multiplicities.

5. Jordan Normal Form (no full proof) and invariant subspaces. Generalised eigenspaces.

6. Dot product, orthogonality(perpendicularity) of vectors, orthogonality of subspaces. Orthogonal complement. Complementary properties of the four fundamental subspaces.

7. Orthogonal projection. Matrix of the orthogonal projection onto the column space of a matrix. Least squares method/linear regression. Geometric intuition, calculus intuition.

8. Orthogonal matrices, isometry. Gram-Schmidt orthogonalisation. Permutation matrices, rotations, reflections. QR-decomposition.

9. LU, PLU, LDU and $PLDU$ decompositions, uniqueness and existence questions.

10. Symmetric matrices, real spectral theorem (Later we called it Principal Axis theorem). Schur's theorem. Five equivalent characterisations of Positive Definite matrices. PSD, ND and NSD matrices. Applications. Sylvester's Law of Intertia for symmetric matrices.

11. Linear function(al)s. Bilinear functions and its matrix in a basis. Change of basis. Perpendicularity($\perp$), left/right perpendicular subspaces, nondegenerate functions. Symmetric and symplectic functions, every function is a sum of two such (if $\mathrm{char}(F) \neq 2$). The function is symmetric or symplectic if and only if $\perp$ is symmetric.

12. Symmetric bilinear functions are diagonalisable. Different diagonalisation algorithms. Sylvester's Law of Intertia over $\mathbb{R}$, signature. Similar statements over $\mathbb{C}$ and $\mathbb{F}_p$.

13. Sesquilinear functions over $\mathbb{C}$, matrix form, adjoint matrix. Hermitian functions. $U \leq V$ is A-invariant if and only if $U^\perp$ is A^*-invariant. Diagonalising Hermitian functions, Sylvester's Law of Intertia.

14. Real and complex Euclidean spaces. Scalar/inner product, length, angle. Cauchy-Schwarz inequality and the triangle inequality. Metric space. Norm.

15. Symmetric/self-adjoint, orthogonal/unitary and normal transformations. Orthogonal eigenspaces, real eigenvalues. Spectral Theorem. Complex ON eigenbasis $\Leftrightarrow$ normal. Equivalent descriptions of unitary transformations

16. Singular values and left/right singular vectors. Singular Value Decomposition over real and complex numbers. Reduced SVD. Uniqueness. Positive (semi)definite case. Geometric interpretation. Polar decomposition, uniqueness. Eckart-Young approximation theorem. Applications.

17. Pseudoinverse, $A^+ : b \mapsto$ leasts square closest solution of $Ax = b$. Special cases of full column rank and full row rank matrices. Pseudoinverse for $A = BC$, full rank factorisation (for example the basis decomposition). Relation to the SVD. Moore-Penrose theorem.

18. Applications. Linear maps to fing special polynomials. Size of finite fields. Hoffman-Singleton theorem. Fibonacci number. Coding theory basics, Hamming distance, Hamming code.