NB, every problem is worth 1 credit notwithstanding the difficulty!

1. Let $R = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix}$. Show that R is left Noetherian but not right Noetherian.

2. Suppose $N \leq M$. Show that M is Noetherian/Artinian if and only if N and M/N are both Noetherian/Artinian.

3. Suppose $R = \bigoplus_{i=1}^k A_i = \bigoplus_{j=1}^n B_j$ with each summand being an ideal that is further indecomposable. Pove that $k = n$ and the two decompositions might differ only in order.

4. Let p be a prime, G a finite p-group and K a field of characteristic p. Verify that $J(KG) = \{\sum \alpha_g g \mid \sum \alpha_g = 0\}$.

5. Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8. Determine the simple modules (irreducible representations) of $\mathbb{R}Q$ and its Wedderburn-Artin decomposition.

6. Let K be an arbitrary field, $T : G \to \mathrm{GL}(n, K)$ a nontrivial irreducible representation. Show that $\sum_{g \in G} T(g) = 0$.

7. Let χ be a character of G. How can we make sense of $\det(\chi)$ as a linear character? Make sure that it is well-defined!

8. *(until 4May)* Show that the degree of every irreducible character divides $|G : Z(G)|$.

9. Let $H \leq G$ and $\mathcal{C}$ a conjugacy class of G. Then $H \cap \mathcal{C} = \cup_i \mathcal{K}_i$ is a partition into H-classes. Pick representatives: $x_i \in \mathcal{K}_i$, $g \in \mathcal{C}$. Show that for any class function χ of H we have

$$\chi^G(g) = |C_G(g)| \sum_i \frac{\chi(x_i)}{|C_H(x_i)|.}$$

10. Let $\mathcal{C}$ be a conjugacy class of G that is not contained in any proper normal subgroup. Further denote $m = |\{\omega_\chi(\hat{\mathcal{C}}) \mid \chi \in \mathrm{Irr}(G)\}|$. Show that every element of G can be written as a product of at most m elements of $\mathcal{C}$.

11. Suppose $A, B \leq G$ are such that $AB = G$. Let χ be a class function of A and φ a class function of G. Prove that

a) $(\chi^G)_B = (\chi_{A \cap B})^B$ and

b) $\varphi \chi^G = (\varphi_A \chi)^G$.

12. We described the irreducible characters of $G \times G$ in terms of those of G. Let us denote $G \cong \Delta(G) = \{(g, g) \mid g \in G\} \leq G \times G$. To each $\eta \in \mathrm{Irr}(G)$ there corresponds a $\Delta(\eta) \in \mathrm{Irr}(\Delta(G))$. Decompose the induced character $\Delta(\eta)^{G \times G}$ into irreducible summands. (The answer is more aesthetic for commutative G.)

13. Determine the character table of D_{2n} using the index 2 cyclic subgroup and character induction. (Do it at least for $n = 5$ and 6.)

14. Let G act doubly transitively on Ω. Suppose $H \leq G$ is such that $|G : H| < |\Omega|$. Show that H is transitive on Ω.

15. Is S_4 and M-group?

16. Suppose that $G = N \rtimes K$ and $\psi \in \mathrm{Irr}(N)$ is a G-invariant *linear* character. Show that it extends to G.

17. A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have corresponding characters α, β and γ. The table below gives the sizes of the conjugacy classes and the values α, β and γ take on them.

	1	21	42	56	24	24
α	14	2	0	-1	0	0
β	15	-1	-1	0	1	1
γ	16	0	0	-2	2	2

Construct the character table of the group.

You may assume, if needed, the fact that $\sqrt{7}$ is not in the field $\mathbb{Q}(\zeta)$, where ζ is a primitive 7th root of unity.

18. A group of order 720 has 11 conjugacy classes. Two representations of this group are known and have corresponding characters α and β, The table below gives the sizes of the conjugacy classes and the values which α and β take on them.

	1	15	40	90	45	120	144	120	90	15	40
α	6	2	0	0	2	2	1	1	0	-2	3
β	21	1	-3	-1	1	1	1	0	-1	-3	0

Prove that the group has an irreducible representation of degree 16 and write down the corresponding character on the conjugacy classes.

19. Let p be a prime, K an algebaically closed field of characteristic p and $G = \langle a, b \rangle$ and elementary Abelian p-group. Determine the endomorphism ring of the KG-module V_{2n} and thus show that it is indecomposable.

$$\dim(V_{2n}) = 2n,\ V_{2n} = \langle v_1, \ldots, v_n, w_1, \ldots w_n \rangle;$$

$$v_i a = v_i + w_i,\ w_i a = w_i b = w_i\ (1 \le i \le n),\ v_i b = v_i + w_{i+1}\ (1 \le i \le n-1),\ v_n b = v_n.$$

20. Let V be a KG-module and $J = J(KG)$ the Jacobson radical. Show that $\mathrm{rad}^i(V) = V \cdot J^i$ and find a similar characterisation of $\mathrm{soc}^i(V)$. Prove that $\min\{i : \mathrm{rad}^i(V) = 0\} = \min\{i : \mathrm{soc}^i(V) = V\}$, which is called the Loewy-length of the module.

21. What can be the Sylow 2-subgroup of a group with character table

φ_1	1	1	1	1	1	1	1
φ_2	1	1	1	ω	ω^2	ω	ω^2
φ_3	1	1	1	ω^2	ω	ω^2	ω
φ_4	3	3	-1	0	0	0	0
φ_5	2	-2	0	-1	-1	1	1
φ_6	2	-2	0	$-\omega$	$-\omega^2$	ω	ω^2
φ_7	2	-2	0	$-\omega^2$	$-\omega$	ω^2	ω

Here ω is a primitive 3-rd root of 1.