Intro to Algebra 2 **Topics** **Spring 2023**

1. Linear maps, transformations, $\mathrm{Hom}(V_1, V_2)$, $\mathrm{Hom}(V)$. Properties, kernel, image. Description of φ by images of basis vectors. Isomorphism, dimension.

2. Homomorphism theorem: If $\varphi : V_1 \to V_2$ then $\dim(\mathrm{Ker}(\varphi))+\dim(\mathrm{Im}(\varphi)) = \dim(V_1)$. $\mathrm{Hom}(V_1, V_2)$ is a vector space, $\mathrm{Hom}(V)$ is a ring (an F-algebra). Projections.

3. Matrix of a linear map/transformation. Multiplication of matrices and maps. Dimension of $\mathrm{Hom}(V_1, V_2)$, $\mathrm{Hom}(V)$. Change of basis, similarity of matrices in $M_{n,k}(F)$ and $M_n(F)$.

4. Eigenvector, eigenvalue, eigenspace. Eigenbasis, diagonalisability. $f(\varphi)$ and $f(A)$ for $f \in F[x]$, $\varphi \in \mathrm{Hom}(V)$ and $A \in M_n(F)$. Minimal polynomial and its roots.

5. Characteristic polynomial of A and of φ and its roots. Cayley-Hamilton Theorem. Algebraic and geometric multiplicities.

6. Jordan Normal Form and invariant subspaces. Generalised eigenspaces.

7. Bilinear functions and its matrix in a basis. Change of basis. Perpendicularity($\perp$), left/right perpendicular subspaces, nondegenerate functions. Symmetric and symplectic functions, every function is a sum of two such (if $\mathrm{char}(F) \neq 2$). The function is symmetric or symplectic if and onply if $\perp$ is symmetric.

8. Symmetric functions are diagonalisable. Different diagonalisation algorithms. Sylvester's Law of Intertia over $\mathbb{R}$, signature. Similar statements over $\mathbb{C}$ and $\mathbb{F}_p$.

9. Sesquilinear functions over $\mathbb{C}$, matrix form, adjoint matrix. Hermitian functions. $U \leq V$ is A-invariant if and only if $U^\perp$ is A^*-invariant. Diagonalising Hermitian functions, Sylvester's Law of Intertia.

10. Real and complex Euclidean spaces. Scalar/inner product, length, angle. Cauchy-Schwarz inequality and the triangle inequality. Metric space. Norm.

11. Symmetric/self-adjoint, orthogonal/unitary and normal transformations. Orthogonal eigenspaces, real eigenvalues. Spectral Theorem. Complex ON eigenbasis $\Leftrightarrow$ normal. Equivalent descriptions of unitary transformations

12. Principal Axis Theorem, Real ON eigenbasis $\Leftrightarrow$ symmetric. Equivalent descriptions of orthogonal transformations. In paricular, block-diagonal decomposition.

13. Quadratic forms (mostly real). Positive/Negative (Semi)Definiteness. Square root of PSD forms and other equivalent conditions. LU and PLU decompositions, existence. Basis (full rank) decomposition. Square root of PD forms and other equivalent conditions. Determining definiteness with (leading) principal minors.

14. Distance of sets in $\mathbb{R}^n$. Orthogonal projection and best approximation. Nullspace, row space and column space of A and of A^TA. Optimal/least square solution of $Ax = b$. Full column rank case: the optimal solution is in the row space. Linear regression.

15. Pseudoinverse $b \mapsto$ leasts square closest solution of $Ax = b$. It is linear and enough to determine on $\mathcal{C}(A)$ and $\mathcal{C}(A)^{\perp}$. Pseudoinverse of full rank matices, and if $A = BC$ a full rank factorisation. Moore-Penrose theorem.

16. Singular values and left/right singular vectors. Modulus of A and its eigenvalues. Singular Value Decomposition over real and complex numbers. Reduced SVD. Uniqueness. Positive (semi)definite case. Geometric interpretation. Relation to the pseudoinverse. Polar decomposition, uniqueness.

17. Matrices coming from graphs, economic models and Markov chains. Incidence matrix and adjacency matrix of a graph. Voltage and conductancy, Kirchhoff's and Ohm's Laws, Euler's Theorem. Nonnegative inverse of $I - A$.

18. Nonnegative matrices coming from dynamic models. Spectrum, spectral radius $\varrho(A)$. Irreducibility. Perron's Lemma. Perron-Frobenius theorem. Lower and upper approximation of $\varrho(A)$ of nonnegative matrices. Cheeger's constant and the spectral gap. Coding theory basics, Hamming distance, Hamming code.