Group Theory **Problem sheet** **Spring 2022**

NB, every problem is worth 1 credit notwithstanding the difficulty!

1. *(until 1March)* Determine the order of G, the isomorphy group of the regular icosahedron.

2. *(until 1March)* How many elements are there of the group which permutes a signed set of $2n$ letters? That is, $\Omega = \{\pm x_1, \pm x_2, \ldots, \pm x_n\}$ and $G = \{\pi \in \mathrm{Sym}(\Omega) \mid x_i\pi = \varepsilon x_j \Leftrightarrow -x_i\pi = -\varepsilon x_j \forall i, j, \varepsilon\}$.

3. Describe the structre of the group of isometries of the regular icosahedron!

4. Dertermine the structure of the group in problem 2!

5. *(until 8March)* Show that if G is a simple group of order $|G| \leq 100$ then $|G| = 60$.

6. *(until 8March)* Show that if G is simple of order 60 then $G \cong A_5$.

7. *(until 16March)* Determine the commutator subgroup of $GL_n(K)$. The result will depend on K and n, of course.

8. *(until 16March)* Determine the maximal subgroups of A_5 (up to conjugacy). For which n is there a primitive subgroup of S_n isomorphic to A_5?

9. *(until 31March)* Is there in $\mathrm{SL}(2,5)$ a subgroup isomorphic to A_5?

10. *(until 31March)* Show that the Sylow 2-subgroup of $\mathrm{SL}(2,3)$ is normal and isomorphic to Q, the quaternion group.

11. (Gaschütz) Suppose p is a prime and V is an Abelian normal p-subgroup of the finite group G. Pick $P \in \mathrm{Syl}_p(G)$. Prove that complement of V in G exists if and only if a complement of V in P exists.

12. Let $G = H \ltimes N$ with $(|H|, |N|) = 1$. Show that $N_G(H) = H \times C_N(H)$. Further, let p be a prime divisor of $|N|$ and show that there exists a H-invariant Sylow p-subgroup of N.

13. *(until 7April)* Suppose $G = AB$ is finite where $A, B \leq G$. If $S \leq G$ is a p-subgroup which contains a Sylow p-subgroup of A and of B then $S \in \mathrm{Syl}_p(G)$ and $S = (S \cap A)(S \cap B)$.

14. *(until 7April)* Let P be a Sylow p-subgroup of a finite group G. Show that if $x, y \in C_G(P)$ are conjugate then the conjugating element can be taken from $N_G(P)$.

15. *(until 1May)* Let G be a finite solvable group, $C = C_G(F(G))$ and $Z = Z(F(G)) = C \cap F(G)$. Show that $O_p(C/Z)$ is trivial for every prime p, and that hence $C = Z$. (In other words, $C_G(F(G)) \leq F(G)$.)

16. *(until 1May)* Let F be a free group on the set X. Show that F' consists of those words in X such that the sum of exponents of x is 0 for each $x \in X$ (in other words, the number of occurrencies of x in the word equals the number of occurencies of x^{-1}).

17. *(until 1May)* Let F be a free group. Let $a, b \in F$, then $ab = ba$ if and only if $\exists u \in F$ with $a = u^h$, $b = u^k$ for some $h, k \in \mathbb{Z}$. (i.e., a, b commutes if and only if they are powers of a common element u)