Introduction to Algebra **Example sheet 9** **Autumn 2022**

1. Determine the monic polynomials of lowest degree a) in $\mathbb{C}[x]$ and b) in $\mathbb{R}[x]$, for which i is a double root and 1 is a triple root!

2. Show that if p is an odd prime, then $\Phi_{2p}(x) = \Phi_p(-x)$. Show that this polynomial is irreducible!

3. About the difficulty of proving the irreducibility of cyclotomic polynomials.

(i) Verify that $\Phi_{12}(x) = x^4 - x^2 + 1$.

(ii) Show that $\Phi_{12}(ax + b)$ does not satisfy the conditions of the Schönemann-Eisenstein criterion for any $a, b \in \mathbb{Z}$, $a \neq 0$ and prime p!

(iii) Prove that $\Phi_{12}(x) \pmod{p}$ is reducible in $\mathbb{F}_p[x]$ for every prime p.

(iv) "By hand" show that $\Phi_{12}(x)$ is irreducible in $\mathbb{Z}[x]$!

4. Let α, β and γ be the complex roots of the polynomial $x^3 - 2x^2 + 4x + 6$. What is the monic polynomial which has roots $\alpha + \beta, \beta + \gamma$ and $\gamma + \alpha$?
(Hint: don't compute the roots, use Vieta's formuli!)

5. Let $\alpha_1, \ldots, \alpha_6$ denote the primitive 7-th roots of 1. What is $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_6^2$?

6. Suppose $g_1(y_1, \ldots, y_n), g_2(y_1, \ldots, y_n) \in F[y_1, \ldots, y_n]$ are two multivariate polynomials such that $g_1(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)) = g_2(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n))$. We prove that $g_1 = g_2$.

(i) By taking their difference we can assume $g_2 = 0$, the tivial polynomial.

(ii) In g_1 let $\lambda y_1^{u_1} y_2^{u_2} \cdots y_n^{u_n}$ be the term where $u_1 + \cdots u_n$ is the largest. If there are several such then pick where $u_2 + \cdots + u_n$ is the largest, etc. Then we pick a unique term, unless $g_1 = 0$, too.

(iii) Show that the lexicographically largest term in $g_1(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n))$ is $\lambda x_1^{u_1+u_2+\cdots+u_n} x_2^{u_2+\cdots+u_n} \cdots x_n^{u_n}$ and it is not cancelled. Contradiction.

7. Compute the following determinants with inversion numbers:

$$\text{a)} \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \qquad \text{b)} \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 4 & 0 & 0 \end{vmatrix} \qquad \text{c)} \begin{vmatrix} 0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 5 \end{vmatrix}$$

8. Compute the following determinants with row/column operations:

$$\text{a)} \begin{vmatrix} 3 & 1 \\ 4 & -3 \end{vmatrix} \qquad \text{b)} \begin{vmatrix} 2 & 2 \\ 6 & 9 \end{vmatrix} \qquad \text{c)} \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 5 \\ 5 & 3 & 1 \end{vmatrix} \qquad \text{d)} \begin{vmatrix} 0 & \ldots & \ldots & 0 & 1 \\ 0 & \ldots & 0 & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & \ldots & 0 \end{vmatrix}_{(n \times n)}$$