Introduction to Algebra **Example sheet 7** **Autumn 2022**

1. Find the roots of the following polynomials and decompose into a product of irreducibles over $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{F}_5$!

(i) $2x^3 - 7x^2 + 2$,

(ii) $x^6 - 2x^5 - x^4 + 4x^3 - 5x^2 + 6x - 3$ and

(iii) $x^5 + 1$.

2. How many irreducible factors does the polynomial $-6x^3 + 6x^2 - 12$ have in $\mathbb{Q}[x]$, $\mathbb{Z}[x]$ and $\mathbb{R}[x]$?

3. What is $\gcd(-6x^3 + 6x^2 - 12, 3x^2 - 3x - 6)$ in $\mathbb{Q}[x]$ and in $\mathbb{Z}[x]$?

4. For which integers c does the polynomial $x^3 + 2x^2 + cx + 4$ have a rational root?

5. Consider the polynomial $x^4 - 6x^3 + 9x^2 + 3$. Is it irreducible over $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{F}_2$?

6. Decompose the polynomial $2x^6 - x^5 - 9x^4 + 4x^3 - 6x^2 + 5x + 5$ as a product of irreducibles in $\mathbb{Q}[x]$ and $\mathbb{F}_5[x]$.

7. Show that if $p(x) \in \mathbb{Q}[x]$ is irreducible, then it has no multiple roots in $\mathbb{R}[x]$!

8. Show that if $p(x) \in \mathbb{Z}[x]$ and $a, b \in \mathbb{Z}$, then $a - b | p(a) - p(b)$!

9. Find a polynomial in $\mathbb{Z}[x]$ such that $\{f(-2), f(1), f(3)\} = \{2, 6, 11\}$ (in any order)!