**Introduction to Algebra** **Example sheet 4** **Autumn 2022**

**1.** Divide $x^7 + x^3 + x - 3$ by $x^4 - x^3 - x^2 + 1$.

**2.** Which of the following sets are rings? (If yes, are they commutative, do they have identity, are they [skew-]fields?)

(i) even integers;

(ii) odd integers;

(iii) real numbers of the form $a + b\sqrt{2}$, where $a, b$ are rational numbers;

(iv) positive real numbers with new addition $a \oplus b := ab$ and new multiplication $a \odot b := a^{\log b}$;

(v) polynomials with 0 constant term;

(vi) polynomials such that the sum of the coefficients is 0;

(vii) polynomials of degree at most 100;

(viii) polynomials such that $a_i = 0$ if $i$ is odd;

(ix) subsets of a set $X$ with addition $A + B := (A \cup B) \setminus (A \cap B)$ (this is called *symmetric difference* and denoted by $A \triangle B$) and multiplication $A \cdot B := A \cap B$;

(x) rational numbers with odd denominator;

(xi) rational numbers with denominator a power of 2;

(xii) the real functions;

(xiii) rational numbers with finite decimal expansion.

**3.** Construct all rings of size 2 and 3. Draw the operational tables for addition and multiplication. Which properties of multiplication hold in them?

**4.** Let $R$ be a ring where $xy = 0$ implies $x = 0$ or $y = 0$. Show that here $ab = ac$ implies $b = c$.Such rings are called *zero-dvisor free.*

**5.** Comment on the following “theorem” and “proof” by the obscure Prof. Smiled:

Theorem: For all $n > 3$ integers $\binom{n}{4} \equiv 1 \pmod 4$.

Proof: We know that $\binom{4}{4} = 1$, so this is true for $n = 4$ and we prove that the theorem by induction. Suppose it holds for $n$. We know that $n - 3 \equiv n + 1 \pmod 4$, so

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24} \equiv \frac{(n+1)n(n-1)(n-2)}{24} = \binom{n+1}{4} \pmod 4.$$

So the claim holds for $n + 1$ and the induction goes through.

**6.** Show that if $S$ is a subring with identity of the real numbers then $S \supseteq \mathbb{Z}$.

**7.** Is there a smallest subring of $\mathbb{Q}$ which contains $1/2$?

**8.** Show that for any natural number $a$ the remainder of the division $x^{3a} - 1 : x^2 + x + 1$ is 0.

**9.** Show that for any natural numbers $a, b, c$ we have $x^2 + x + 1 | x^{3a+2} + x^{3b+1} + x^{3c}$.