Introduction to Algebra **Example sheet 2** **Autumn 2022**

1. Prove the following claim using induction. The sum of the first few cubic numbers is always a square.

2. Compute the greatest common divisor of the following pairs of integers or expressions. (Try to do this both with the smallest non-negative remainder and with the remainder of smallest absolute value.) Express it as an integral combination of the original numbers (i.e. as $(a, b) = xa + yb$).

1. 2020, 1974;
2. 89, 55;
3. n, $n + 1$;
4. $2n$, $3n + 1$;

3. At most how many steps does the Euclidean Algorithm need to reach the greatest common divisor of two numbers less then 100? What happens if we use the remainder of smallest absolute value instead of the smallest non-negative one?

4. Show that for three integers a, b, c the linear equation $ax+by = c$ is soluble among the integers if and only if $(a, b)|c$. Show that if $a, b > 0$ and $c > ab$ then the equation is soluble among the *positive* integers.

5. Suppose that $(a, b) = 5$. What can $(a + b, a - b)$ be?

6. Let $X = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \in \{0, 1\}\}$. We define a binary relation $\leq'$ on X by: $(a_1, a_2) \leq' (b_1, b_2)$ if either $a_1 < b_1$ or ($a_1 = b_1$ and $a_2 \leq b_2$). Show that this relation is reflexive, transitive and antisymmetric. We define a unary operation $+'$ on X by: $+'(a_1, a_2) = (a_1, a_2 + 1)$. Show that X is well ordered: every nonempty subset of X has a minimal element, but the principle of induction (with respect to this "+" step) does not hold.

7. Can you define an "addition" and a "multiplication" on X which obeys the identities we have on $\mathbb{N}$?

8. We prove that every human eye is of the same colour.

We use induction on the number of people to consider. If we have only one person, then it is true.

Suppose we have verified that among $n \geq 1$ people all have the same coloured eyes. We now show this for $n + 1 \geq 2$ people. Pick one of them, A, if we remove A from the group the remaining ones have eyes of the same colour by induction. Now pick another person, B (we already know that B has the same colour as the others). If we remove B from the group instead of A, there are again n people, so, by induction, all the remaining have the same eye-colour, even A. So all $n + 1$ have eyes of the same colour.

What is wrong with the proof?

9. Determine all solutions of the following equality, where a, b are positive integers and p is a positive prime.

$$[a, b] + (a, b) = a + b + p$$