Introduction to Algebra **Example sheet 12** **Autumn 2022**

1. Which of the following (with usual operations) are vector spaces over the real numbers?

(i) $\{f \in \mathbb{R}[x] \mid \deg(f) = 100 \text{ or } f = 0\}$;

(ii) $\{f \in \mathbb{R}[x] \mid \deg(f) \leq 100 \text{ or } f = 0\}$;

(iii) $\{f \in \mathbb{R}[x] \mid \deg(f) \geq 100 \text{ or } f = 0\}$;

(iv) polynomials with 0 constant term;

(v) polynomials such that the sum of the coefficients is 0;

(vi) polynomials that give constant remainder divided by $x^2 + 1$;

(vii) polynomials such that $a_i = 0$ if i is odd;

(viii) real series with first term 0;

(ix) real series with all, but finitely many terms being 0;

(x) real series with infinitely many 0 terms;

(xi) rational series;

(xii) matrices with entries 0 below the main diagonal (upper triangular matrices);

(xiii) matrices with entries 0 below or on the main diagonal (strictly upper triangular matrices);

(xiv) upper bounded functions;

(xv) continuous functions;

(xvi) periodic functions;

(xvii) matrices A such that $A = -A^T$ (skew-symmetric matrices);

2. Which of the following subsets of $\mathbb{R}^3$ is a subspace?

a) $\{v \in \mathbb{R}^3 |\ |v| = 1\}$, b) $\{(x, y, z) | x + 2y + z = 0\}$ and c) $\{(x, y, z) | x + 2y + z = 1\}$.

NB. Not every line/plane is a subspace. There are *affine subspaces* of the form $u + W$, where u is a vector and W is a subspace.

3. Verify that
a) the intersection of two affine subspaces is an affine subspace or empty, but
b) the union of two subspaces is a subspace if and only if one contains the other
c) (if the underlying field has more than 2 elements then) the union of two affine subspaces is an affine subspace if and only if one contains the other.

4. Choose a maximal linearly independent system of the columns of the following matrix A! Write the other columns as linear combinations of the previous ones. Compute a basis of $\mathcal{N}(A) = \mathrm{Ker}(A)$ (where $\mathcal{N}(A) = \mathrm{Ker}(A)$ is the kernel/nullspace of A)!

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 2 & -1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 & 3 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix}$$

5. Find a basis in those subsets of problem 1 which are subspaces!

6. Let $B = \{(1,0,1)^T, (0,2,-1)^T, (1,1,0)^T\}$.
a) Show that B is a basis in $\mathbb{R}^3$!
b) What is the coordinate vector of $(1,0,0)^T$ with respect to B?
c) For which vector $w \in \mathbb{R}^3$ is $[w]_B = (5,1,-2)^T$?

7. Are the following matrices invertible? If yes, compute their inverses!

$$A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

8. Solve the following matrix equations (A, B, C and D are as above)!

a) $CX = D$, b) $BX = C$, c) $XB = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 1 \end{pmatrix}$, $XB = A \begin{pmatrix} 1 & 2 & 3 \\ 5 & 3 & 1 \end{pmatrix}$
b) the union of two affine subspaces is an affine subspace if and only if one contains the other

9. (For Amy) Let $R = \mathbb{R}[x]$ be the set of real polynomials. We say that a map $\varphi : R \to R$ is linear if $\varphi(f+g) = \varphi(f) + \varphi(g)$ and $\varphi(\lambda f) = \lambda\varphi(f)$ for all polynomials f, g and real number λ. If φ and ψ are two such lienar maps then $\tau = \varphi + \psi$ and $\sigma = \varphi \circ \psi$ are the sums and products of them: $\tau(f) = \varphi(f) + \psi(f)$ and $\sigma(f) = \psi(\varphi(f))$. With these operations the linear maps form a ring, call it $L(R)$. It has an identity element, $\mathrm{id} \in L(R)$, the identity map. Derivation is also in $L(R)$, call it Der. Show that if $\varphi \in L(R)$ then $\varphi(0) = 0$. Show also that Der has no right inverse but it has infinitely many left inverses.