Intro to Algebra 1 **Topics** **Autumn 2022**

1. Sets, intersection, union, difference, symmetric difference (Δ), Venn diagrams, Cartesian product, cardinality.

2. Relations, arity. Equivalence relation. Equivalence Relation Theorem. Poset.

3. Functions and operations. Domain, codomain, injective, surjective, bijective. Image/range, Kernel (an eq. relation). Operation table, closure of subsets with respect to operations.

4. Integers. Principle of induction, well ordering. Division, division with remainder. Euclid's algorithm, gcd, lcm. Solvability of $(a, b) = ax + by$. Units, irreducible and prime integers, Fundamental Theorem of Arithmetic.

5. Numbers at different bases. Horner's method. Congruences, Diophantine equations. Chinese Remainder Theorem. Complete and reduced systems of residues. Euler's totient function. Wilson's Theorem.

6. Rings. Polynomials over fields and rings. Evaluation map, roots. Divisibility, long division, division with remainder. Root factor. Number theory of polynomials.

7. Multiplicity of roots and testing with the derived polynomial. Degree n polynomial has at most n roots. Vieta's formuli. Symmetric polynomials, elementary symmetric polynomials. Fundamental Theorem of Symmetric Polynomials.

8. Polynomials over the integers. Rational root test, primitive polynomial. Gauss' Lemma (three with this name). Schönemann-Eisenstein criterion.

9. Mod p fields ($\mathbb{F}_p$), $\mathbb{F}_p)[x]$. Order of an element mod n. Euler-Fermat Theorem, existence of primitive root mod p.

10. Complex numbers, complex plane, modulus and argument. DeMoivre formula, roots of 1, order, primitive roots of 1. Cyclotomic polynomials. Irreducibility of $\Phi_p(x)$ over the rationals (special case of Gauss' Theorem).

11. Permutations, inversions, parity. Matrix, operations, transpose, determinant. Elementary (row/column) operations on the determinant.

12. Laplace (cofactor) expansion, skew expansion, adjugate matrix. Vandermonde determinant. Product theorem. Left/right invertibility. Elementary matrices (E_{ij}). Cramer's Rule.

13. Linear systems of equations. Gauss' eliminating method. Pivots, contradictory rows, free parameters, backward substitution. Homogeneous systems, $Ax = b$ notation. Solvbability and unique solvability criteria.

14. Vector spaces. Subspaces. Linear combination, generated subspace, closure conditions. Sum and direct sum. Dependence, independence. Four properties. Infinite sets.

15. Column space of a matrix. Generation and dependence in the language of systems of linear equations.

16. Rank of a matrix. Interpretation in terms of the system of equations and in terms of the vector spaces of row/column vectors.

17. Basis. Replacement theorem, existence of basis if the vector space is finitely generated. Dimension. Zorn's Lemma, basis and dimension of infinitely generated vector spaces.