Algebra 1 BSc   Solutions to FT problems   2021 Autumn

4. Determine $\Phi_{10}(x)$ and $\Phi_{15}(x)$ and their greatest common divisor.

Use $x^n = \Pi_{d|n}\Phi_d(x)$ *to get* $\Phi_{10}(x) = \frac{x^{10}-1}{(x-1)(x^4+x^3+x^2+x+1)(x+1)} = x^4 - x^3 + x^2 - x + 1$ *and* $\Phi_{15}(x) = \frac{x^{15}-1}{(x-1)(x^4+x^3+x^2+x+1)(x^2+x+1)} = \frac{x^{10}+x^5+1}{x^2+x+1} = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$. *The greatest common divisor is* 1 *because they are irreducible rational polynomials.*

5. How many Abelian groups exist of order 200 up to isomorphism?

By the Fundamental Teorem of Finite Abelian Groups, such a group is the direct product of its Sylow 2*-subgroup of order* 8 *and its Sylow* 5*-subgroup of order* 25*. Which are in turn direct products of cyclic groups.* $8 = 2\cdot2\cdot2 = 4\cdot2 = 8$*, three possibilities:* $Z_2 \times Z_2 \times Z_2$, $Z_4 \times Z_2$, Z_8*, while* $25 = 5 \cdot 5 = 25$*, two possibilities:* $Z_5 \times Z_5$, Z_{25}*, so altogether* $3 \cdot 2 = 6$ *possibilities. One of them is* $Z_4 \times Z_2 \times Z_{25}$.

6. Let E denote the algebraic closure of $\mathbb{F}_2$, the field of two elements. What is its characteristic char(E)? What is $\dim_{\mathbb{F}_2} E$? Show that for every $n \geq 1$ there exists a subfield $E_n \leq E$ such that $|E_n| = 2^n$.

$1 + 1 = 0$ *in* E *because* $1 \in F$ *and there this euqality holds, so the characteristic is* 2*. As* E *contains all the roots of polynomials over* F *of arbitrary degree. (You need that such a polynomial exists without multiple roots,* $x^{2k+1} + 1$ *is such for every* k*.) So* E *is infinite, hence its dimension over* F *is infinite. In particular, it contains all the roots of* $x^{2^n} - x$*, which form a (the) field of* 2^n *elements.*