Algebra 1 **Topics** **Autumn 2021**

1. Groups, basic properties, examples, cyclic groups

2. Order of an element, set product, Cayley table

3. Subgroup, proper subgroup. Cyclic groups containing each other. Quaternion groups, dihedral groups

4. Generated subgroup, two descriptions, generating set, examples

5. Symmetric group, product of disjoint cycles, product of transpositions. Special generating sets: $(1\ i)$ transpositions, $(i\ i+1)$ transpositions, 2-element generating set. Odd and even permutations, A_n

6. Cosets of a subgroup, index, Lagrange's theorem, order of $g \in G$ divides the order of G

7. Normal subgroup, conjugation, conjugacy class, conjugate subgroup. Normaliser, centraliser, centre

8. Conjugation in S_n, restriction of conjugacy classes to A_n. A_n is simple

9. Homomorphism, kernel, image. Mono, epi, iso. Factor group, natural homomorhism, homomorphism theorem. Automorphism, inner automorphism. Isomorphism theorems. Characteristic subgroup

10. Direct product, outer/inner, unique expression. Direct product with several factors. Fundamental theorem of finite Abelian groups.

11. Cauchy's theorem. Permutation groups and group actions on a set. Orbit, stabiliser, transitivity, regularity

12. Sylow's theorems, applications. Finite and infinite p-groups. Centre, normal subgroups and maximal subgroups of finite p-groups

13. Commutator, derived subgroup, description. Normal chain, solvable group. Higher commutation subgroups, derived series, derived length. Dependency between the derived lengths of a group and subgroups, factor groups of it. Composition chain, composition factors, Jordan-Hölder theorem. Solvability theorems of Burnside, Zassenhaus and Feit-Thompson

14. Free group, presentation (generators and defining relations). Dyck's theorem, examples

15. Linear groups (GL, SL, PGL, PSL), Dickson's theorem, order of these linear groups over finite fields

16. Rings, basic properties, examples. If $|R| > 1$ and for $a \neq 0$ and for every b the equation $ax = b$ is solvable in R then R is a skewfield

17. Subring, ideal (left/right ideal), generated ideal, principal ideal. Simple ring, examples. Boole ring, set ring, group ring, group algebra

18. Homomorphism, factor ring, homomorphism theorem, isomorphism theorems

19. Integral domain, field of fractions. Divisibility and ideals. Unit, associates, primes irreducibles. UFD and its description using principal ideals. PID, ED. Greatest common divisor and its expression, examples

20. Characteristic of a field, prime field. Field extension, algebraic and transcendental elements, their charcterisations. Degree of an algebraic element and of a field extension. Algebraic extension, degree theorem. Algebraic elements for a subfield. Abstract extension with a root of an irreducible polynomial. Splitting field, abstract construction. Algebraically closed field, algebraic closure, existence and uniqueness

21. Finite fields, construction. Berlekamp's algorithm for factorisation of polynomials over $\mathbb{F}_p$.

22. Cylotomic polynomials, Gauss' proof of their irreduicibility. Wedderburn's theorem. Finite subgroup of a field is cyclic.