Distributive biracks

Přemysl Jedlička[1], Agata Pilitowska[2], Anna Zamojska-Dzienio[2]

[1]Department of Mathematics, Faculty of Engineering, Czech University of Life Sciences,
[2]Faculty of Mathematics and Information Science, Warsaw University of Technology

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Motivation

Quandles and racks (Joyce 1982, Matveev 1984) $\leftrightarrow$ (classical) knot theory.

Biquandles and biracks (Fenn, Jordan-Santana, Kauffman 2004) $\leftrightarrow$ virtual knot theory (Kauffman 1999).

There is a one-to-one correspondence between biracks and non-degenerate set-theoretical solutions of the Yang-Baxter equation.

Equational characterization

Definition (Stanovský 2006)

A structure $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ with four binary operations is called a **birack**, if the following holds for any $x, y, z \in X$:

$$\begin{aligned}
x \circ (x \backslash_\circ y) = y &= x \backslash_\circ (x \circ y),\\
(y /_\bullet x) \bullet x = y &= (y \bullet x) /_\bullet x,\\
x \circ (y \circ z) &= (x \circ y) \circ ((x \bullet y) \circ z),\\
(x \circ y) \bullet ((x \bullet y) \circ z) &= (x \bullet (y \circ z)) \circ (y \bullet z),\\
(x \bullet y) \bullet z &= (x \bullet (y \circ z)) \bullet (y \bullet z).
\end{aligned}$$

Multiplication groups

In a birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$:

left translations: $L_x \colon X \to X$ by x; $L_x(a) = x \circ a$ and
right translations: $\mathbf{R}_x \colon X \to X$ by x; $\mathbf{R}_x(a) = a \bullet x$
are **bijections**.

Three types of multiplication groups:

$$\mathrm{LMlt}(X) = \langle L_x : \ x \in X \rangle$$
$$\mathrm{RMlt}(X) = \langle \mathbf{R}_x : \ x \in X \rangle$$
$$\mathrm{Mlt}(X) = \langle L_x, \ \mathbf{R}_x : \ x \in X \rangle$$

Biracks vs. racks

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be a birack, where for every $x, y, \in X$:

$$x \circ y = y.$$

Then $(X, \bullet, /_\bullet)$ is a (right) rack.

A rack and a **derived solution** are exactly the same.

Distributive biracks

A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is **left distributive**, if for every $x, y, z \in X$:

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z),$$

and it is **right distributive**, if for every $x, y, z \in X$:

$$(y \bullet z) \bullet x = (y \bullet x) \bullet (z \bullet x).$$

The birack is **distributive** if it is left and right distributive.

Examples

Permutational birack (Lyubashenko (Drinfeld 1992)):

$X \neq \varnothing$, $f, g\colon X \to X$ bijections such that $fg = gf$.

Define operations:

$$x \circ y = f(y),\ x \backslash_\circ y = f^{-1}(y),$$
$$x \bullet y = g(x),\ x /_\bullet y = g^{-1}(x).$$

Then $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is a **permutational birack**.

If $f = g = \mathrm{id}$, the birack is a **projection** one.

Examples

Derived biracks

$(X, \circ, \backslash_\circ)$ left rack.
Define operations $\bullet, /_\bullet \colon X \times X \to X$ as $x \bullet y = x = x /_\bullet y$. Then the structure $\mathbf{B}_L(X, \circ, \backslash_\circ) = (X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is a **left derived birack**.

Symmetrically, $\mathbf{B}_R(Y, \bullet, /_\bullet)$ is a **right derived birack**.

For a left rack $(X, *, \backslash_*)$ and a right rack $(Y, \triangle, /_\triangle)$, the product $\mathbf{B}_L(X, *, \backslash_*) \times \mathbf{B}_R(Y, \triangle, /_\triangle)$ is a distributive birack with $\mathrm{Mlt}(X \times Y) \cong \mathrm{LMlt}(X) \times \mathrm{RMlt}(Y)$.

Examples

Wada switch

$(G, \cdot, e)$ a group. Define operations:

$$x \circ y = xy^{-1}x^{-1},\ x\backslash_\circ y = x^{-1}y^{-1}x,$$
$$x \bullet y = xy^2,\ x/_\bullet y = xy^{-2}.$$

The birack $(G, \circ, \backslash_\circ, \bullet, /_\bullet)$ known as the **Wada switch** or **Wada biquandle** (Fenn et al.).

$(G, \circ, \backslash_\circ, \bullet, /_\bullet)$ is left distributive iff $y^2 \in Z(G)$, for all $y \in G$, and is right distributive iff $x^4 = e$ and $x^2 \in Z(G)$, for all $x \in G$.

Alternative characterization

Proposition

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *be a structure with four binary operations. Then* $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is a distributive birack if and only if the following conditions are satisfied*

(i) $(X, \circ, \backslash_\circ)$ *is a left rack and* $(X, \bullet, /_\bullet)$ *is a right rack,*

(ii) *for all* $x, y, z \in X$

$$\begin{aligned}
(x \bullet y) \circ z &= x \circ z,\\
x \bullet (y \circ z) &= x \bullet z,\\
x \circ (y \bullet z) &= (x \circ y) \bullet z.
\end{aligned}$$

Multiplication groups

Proposition

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ be a distributive birack. Then, for each $x \in X$, the bijections L_x and $\mathbf{R}_x$ are automorphisms of $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$.

Groups $\mathrm{LMlt}(X)$, $\mathrm{RMlt}(X)$, $\mathrm{Mlt}(X)$ are normal subgroups of the automorphism group of a distributive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$.

Involutive distributive biracks

A birack is **involutive** if it satisfies for every $x, y \in X$:

$$(x \circ y) \circ (x \bullet y) = x \quad \Leftrightarrow \quad x \bullet y = (x \circ y) \backslash_{\circ} x,$$
$$(x \circ y) \bullet (x \bullet y) = y \quad \Leftrightarrow \quad x \circ y = y /_{\bullet} (x \bullet y).$$

1. in involutive biracks, $\mathrm{LMlt}(X) = \mathrm{RMlt}(X) = \mathrm{Mlt}(X)$,
2. an involutive birack is left distributive iff it is right distributive,
3. involutive distributive biracks correspond to 2-reductive racks (Agata's talk).

Retraction relations

$(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ a birack.

Three retraction relations:

$$
\begin{aligned}
a \sim b \quad &\Leftrightarrow \quad L_a = L_b \quad \Leftrightarrow \quad \forall x \in X \quad a \circ x = b \circ x, \\
a \backsim b \quad &\Leftrightarrow \quad \mathbf{R}_a = \mathbf{R}_b \quad \Leftrightarrow \quad \forall x \in X \quad x \bullet a = x \bullet b, \\
a \approx b \quad &\Leftrightarrow \quad a \sim b \wedge a \backsim b \quad \Leftrightarrow \quad L_a = L_b \wedge \mathbf{R}_a = \mathbf{R}_b.
\end{aligned}
$$

They are congruences for a distributive birack.

retract of X: $\mathrm{Ret}(X) = (X/\approx, \circ, \backslash_\circ, \bullet, /_\bullet)$.

iterated retraction: $\mathrm{Ret}^0(X) = (X, \circ, \backslash_\circ, \bullet, /_\bullet)$ and $\mathrm{Ret}^k(X) = \mathrm{Ret}(\mathrm{Ret}^{k-1}(X))$, for any natural number $k > 1$.

(Multi)permutational distributive birack

A birack is of **multipermutation level** k, if $|\mathrm{Ret}^k(X)| = 1$ and $|\mathrm{Ret}^{k-1}(X)| > 1$.

Gateva-Ivanova 2018: an involutive birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is of multipermutation level k if and only if it is left k-permutational.

Let $k \in \mathbb{N}$. A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is:

1. **left k-permutational**, if for every $x, y, x_1, x_2, \ldots, x_k \in X$:

$$(\ldots((x \circ x_1) \circ x_2) \ldots) \circ x_k = (\ldots((y \circ x_1) \circ x_2) \ldots) \circ x_k.$$

Non-involutive distributive biracks

Let $k \in \mathbb{N}$. A birack $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ is:

1. **right k-permutational**:

$$x_0 \bullet (\dots (x_{k-2} \bullet (x_{k-1} \bullet x)) \dots) = x_0 \bullet (\dots (x_{k-2} \bullet (x_{k-1} \bullet y)) \dots),$$

2. **left k-reductive**:

$$(\dots ((x_0 \circ x_1) \circ x_2) \dots) \circ x_k = (\dots ((x_1 \circ x_2) \circ x_3) \dots) \circ x_k,$$

3. **right k-reductive**:

$$x_0 \bullet (\dots (x_{k-2} \bullet (x_{k-1} \bullet x_k)) \dots) = \\ x_0 \bullet (\dots (x_{k-3} \bullet (x_{k-2} \bullet x_{k-1})) \dots).$$

Main Theorem

Theorem

Let $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *be a distributive birack and let* $k \geq 2$. *Then the following conditions are equivalent:*

(i) $|\mathrm{Ret}^k(X)| = 1$,

(ii) $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is* k*-reductive,*

(iii) $(X, \circ, \backslash_\circ, \bullet, /_\bullet)$ *is* k*-permutational,*

(iv) $\mathrm{Mlt}(X)$ *is nilpotent of class at most* $k-1$.

THANK YOU!