# Generalised transversals of Latin squares

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[EJC1] I. M. Wanless, A generalisation of transversals for Latin squares, *Electron. J. Combin.*, 9(1) (2002), #R12.

[EJC2] N. J. Cavenagh and I. M. Wanless, Latin squares with no transversals, *Electron. J. Combin.* 24(2) (2017), #P2.45.

# Plexes in Latin squares

A *k-plex* in a Latin square of order $n$ is a selection of $kn$ entries, with $k$ in each row and column and $k$ of each symbol.

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| 2 | 1 | 4 | 3 | 6 | 5 |
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**Conjecture:** [Ryser] Every LS of odd order has a transversal.

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**Conjecture:** For all even $n > 4$ there is a LS($n$) with a triplex but no transversals.

# Protoplexes

A protoplex is a partial LS with $k$ filled cells in each row and column, and $k$ occurrences of each symbol $1, 2, \ldots, n$.

$$\begin{pmatrix} \cdot & \cdot & 2 & 3 & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 4 & 2 \\ \cdot & \cdot & 4 & \cdot & 3 \\ 1 & 0 & \cdot & \cdot & \cdot \end{pmatrix}$$

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**Theorem:** [Barber/Kühn/Lo/Osthus/Taylor'17] True for $k < (\frac{1}{25} - \epsilon)n$.

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**Theorem:** [Egan/W.'11] For any proper divisor $k$ of $n$ there is a LS which partitions into *indivisible* $k$-plexes.

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**Lemma:** Let $K$ be a $k$-plex in a LS of order $n$.

$$\sum_{(r,c,s)\in K} \Delta(r, c, s) \mod n = \begin{cases} 0 & \text{if } k \text{ is even or } n \text{ is odd,} \\ n/2 & \text{if } k \text{ is odd and } n \text{ is even.} \end{cases}$$

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In fact, if we replace the last $\lfloor \sqrt{n} \rfloor$ rows of $\mathbb{Z}_n$ with *any* other choice of rows, there will still be no transversals. This is because the $\Delta$ values don't change by much.

# Example

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \\ 2 & 3 & 4 & 5 & 6 & 7 & 0 & 1 \\ 3 & 4 & 5 & 6 & 7 & 0 & 1 & 2 \\ 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 7 & 6 & 1 & 0 & 3 & 2 & 5 & 4 \\ 6 & 0 & 7 & 1 & 2 & 4 & 3 & 5 \\ 5 & 7 & 0 & 2 & 1 & 3 & 4 & 6 \end{bmatrix}$$

Which has these $\Delta$ values:

$$\begin{bmatrix} 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & 1 & -1 & \cdot \\ -2 & -1 & -1 & \cdot & -2 & -1 & -1 & \cdot \end{bmatrix}$$

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**Theorem:** For all even $n > 4$ there is a LS of order $n$ that contains a 3-plex but no transversal.

# The main idea

$Z_{12}$ contains these entries:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | | | | | | | |
| | | | | | 6 | 7 | 8 | | | | |
| | | | | | | | | 10 | 11 | 0 | |
| | | | | | | | | | | | 2 |
| 4 | 5 | 6 | | | | | | | | | |
| | | | 8 | 9 | 10 | | | | | | |
| | 7 | | | | | 0 | 1 | | | | |
| | | 9 | | | | | 2 | 3 | | | |
| | | | 11 | | | | | 4 | 5 | | |
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| 4 | 5 | 6 | | | | | | | | | |
| | | | 8 | 9 | 10 | | | | | | |
| | 7 | | | | | 0 | 1 | | | | |
| | | 9 | | | | | 2 | 3 | | | |
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- We still can't prove that every LS is divisible! (Beware: [Egan'11] has shown that for all $n > 3$ there are LS($n$) that split into two indivisible plexes.)
- Are there LS that have an $a$-plex and a $c$-plex but no $b$-plex for odd $a < b < c$?

**Conjecture:** For each odd $k$ there exists $N$ such that for all even $n \geqslant N$ there exists a latin square of order $n$ that contains a $k$-plex but no $k'$-plex for odd $k' < k$.