ON TOTAL MULTIPLICATION GROUPS OF LOOPS

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LOOPS 2019
July 7–13, 2019
Budapest University of Technology and Economics, Hungary

Multiplication groups of loops are a standard tool of algebraic loop theory. It does not seem likely that the total multiplication groups will achieve a similar status .... Nevertheless, there are situations when middle translations cannot be avoided, for example, when considering the connection to paratopic (isostrophic) loops.

The purpose of this work is to consider some basic facts on the total multiplication groups, by analogy with the known ones on the usual multiplication groups, in particular, to consider the total multiplication groups of some known classes of loops, such as IP-loops, Moufang loops and Bol loops.

It is well known that the notions *multiplication group* and *inner mapping group* of a quasigroup (loop) were introduced by Albert in [Quasigroups I, 1944] and, respectively, by Bruck in [A survey of binary systems, 1958].

Both, the multiplication group and the inner mapping group are invariant (up to isomorphism) under the isotopy of loops.

Later, in 1969 [4], Belousov considered the group generated by all left, right and middle translations of a quasigroup, which he called the "complete multiplication groups", and remarked that such groups are invariant under the parastrophy of quasigroups and play a similar role in defining normal subquasigroups.

Let $(Q, \cdot)$ be a quasigroup. Denote:

$Mlt(Q, \cdot) =< L_x, R_x | x \in Q >$ - the multiplication group,
$TMlt(Q, \cdot) =< L_x, R_x, D_x | x \in Q >$ - the total multiplication group,
where $L_x(y) = x \cdot y, \;\; R_x(y) = y \cdot x, \;\; D_x(y) = y \backslash x, \forall x, y \in Q$.

If Q is a loop with the unit e, then

$Inn(Q) = (Mlt(Q))_e$ (the stabilizer of e) - the inner mapping group of Q;
$TInn(Q) = (TMlt(Q))_e$ - the total inner mapping group of Q.

The case of $(RIP, LIP)IP-$ loops ($D_x = R_x I$, where $I(x) = x^{-1}$)

Proposition 1. If Q is an $IP-$ loop (a group) then:
(i) $Mlt(Q)$ is a normal subgroup of index two of the group $TMlt(Q)$;
(ii) $TMlt(Q) = Mlt(Q) \rtimes < I >$, where $I : Q \mapsto Q, I(x) = x^{-1}$;

Proposition 2. If Q is an $RIP-$loop or an $LIP-$loop then $TMlt(G) =< Mlt(Q), I >$.

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The behavior of the total multiplication groups under the isostrophy of loops

Definition. Two quasigroups are called isostrophic (principally isostrophic) if they have parastrophes that are isotopic (principally isotopic).

Using the fact that every quasigroup operation has six parastrophes (some of them may coincide)we get:

Theorem 1

If $(Q,\cdot)$ and $(Q,\circ)$ are isostrophic (principally isostrophic) loops than:
(i) $TMlt(Q,\cdot) \cong TMlt(Q,\circ)$ (resp. $TMlt(Q,\cdot) = TMlt(Q,\circ)$);
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Generators of the (total) inner mapping groups

1. $Inn_h(Q) =< L_{x,y}; R_{x,y}; T_x | x, y \in Q >$, where
$L_{x,y} = L^{-1}_{x \circ y} L_x L_y$, $R_{x,y} = R^{-1}_{x * y} R_y R_x$, $T_x = L^{-1}_{\sigma(x)} R_x$,
$x \circ y = R_h^{-1}(a \cdot R_h b)$, $x * y = L_h^{-1}(L_h \cdot b)$, $\sigma = R_h^{-1} L_h$;
2. If $(Q, \cdot)$ is a loop with the unit e, then the generators of $Inn_e(Q) = Inn(Q)$ are:
$L_{x,y} = L^{-1}_{x \cdot y} L_x L_y$, $R_{x,y} = R^{-1}_{x \cdot y} R_y R_x$, $T_x = L_x^{-1} R_x$.

3. Let $(Q, \cdot)$ be a quasigroup. Belousov proved in [4] that the group $TInn_h(Q)$ is generated by the set of mappings

$$L^{(\cdot)}_{x,y}, R^{(\cdot)}_{x,y}, T^{(\cdot)}_x, N^{(\cdot)}_{x,y}, S^{(\cdot)}_x,$$

where $N^{(\cdot)}_{x,y} = L^{(/)}_{x,y}$, $S^{(\cdot)}_x = T^{(/)}_x$, for $\forall x, y \in Q$.

In fact, at present there are known several sets of generators of the total inner mapping group.

A set of generators of the total inner mapping group of a loop

Theorem 2

If $(Q, \cdot)$ is a loop, then

$$TInn(Q) =< L_{x,y}; R_{x,y}; T_x; P_{x,y}; U_x | x, y \in Q >,$$

where $L_{x,y} = L_{x \cdot y}^{-1} L_x L_y$, $R_{x,y} = R_{x \cdot y}^{-1} R_y R_x$, $T_x = L_x^{-1} R_x$, $P_{x,y} = R_y^{-1} L_x D_y D_x$, $U_x = D_x R_x$.

Corollary 1

If $(Q, \cdot)$ is a power associative loop, then

$$TInn(Q) =< L_{x,y}; R_{x,y}; P_{x,y}; U_x | x, y \in Q >,$$

where $L_{x,y} = L_{x \cdot y}^{-1} L_x L_y$, $R_{x,y} = R_{x \cdot y}^{-1} R_y R_x$, $P_{x,y} = R_y^{-1} L_x D_y D_x$, $U_x = D_x R_x$.

Corollary 2

If $(Q, \cdot)$ is a middle Bol loop, then

$$TInn(Q) =< R_{x,y}; P_{x,y}; U_x | x, y \in Q > .$$

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Analog properties to that of $Mlt(Q)$ and $Inn(Q)$

Proposition 3. ($\xi : x \mapsto L_x TInn(Q)$ is a bijection)

Let $(Q, \cdot)$ be a quasigroup and $h \in Q$. The following statements hold:
1. If $TInn_h(Q) = \{\varepsilon\}$ then $(Q, \cdot)$ is an abelian group;
2. If $(Q, \cdot)$ is a finite quasigroup, then
$$|TMlt(Q)| = |Q||TInn_h(Q)|.$$

Proposition 4. ($\varphi : P_{x_1}^{\varepsilon_1} \dots P_{x_n}^{\varepsilon_n} \mapsto P_{\theta x_1}^{\varepsilon_1} \dots P_{\theta x_n}^{\varepsilon_n}$)

Let $\theta : Q \mapsto G$ be a surjective homomomorphism of loops, from $(Q, \cdot)$ to $(G, \circ)$. Then θ induces a homomorphism $\varphi : TMlt(Q) \mapsto TMlt(G)$, such that $\varphi(\alpha)\theta = \theta\alpha, \forall \alpha \in TMlt(Q)$, and the following statements hold:
1. $TMlt(Q/H) \cong TMlt(Q)/H^*$;
2. $TInn(Q/H) \cong TInn(Q)/(H^* \cap TInn(Q))$,

where $H^* = \{\alpha \in TMlt(Q) | H(\alpha(x)) = Hx, \forall x \in Q\}$, $H = Ker\varphi$.

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The center of the total multiplication group

Let $(Q,\cdot)$ be a loop. The *nucleus*, *commutant* and the *centre* of $(Q,\cdot)$ are defined, respectively, as follows:
$N = \{a \in Q | ax \cdot y = a \cdot xy, xa \cdot y = x \cdot ay, xy \cdot a = x \cdot ya, \forall x, y \in Q\}$
$C = \{a \in Q | ax = xa, \forall x \in Q\}$ and $Z(Q) = N \cap C$.

Theorem 3

If $(Q,\cdot)$ is a loop with the unit e, then
1. $Z(TMlt(Q,\cdot)) = \mathcal{L}_{(\cdot)} \cap \mathcal{R}_{(\cdot)} \cap \mathcal{F}'_{(\cdot)} = \{\varphi \in Z(Mlt(Q,\cdot)) | \varphi^2 = \varepsilon\} = \{L_x | x \in Z(Q), x^2 = e\}$, where $\mathcal{F}'_{(\cdot)} = \{\varphi \in \mathcal{F}_{(\cdot)} | \varphi^* = \varphi^{-1}\} \leq \mathcal{F}_{(\cdot)}$;
2. $Z(TMlt(Q))e \trianglelefteq Q$;
3. $Z(TMlt(Q))e = \{a \in Z(Q) | a^2 = e\} = \{a \in Q | \varphi(a) = a, \forall \varphi \in TInn(Q,\cdot)\}$.

Theorem 4

If $(Q,\cdot)$ and $(Q,\circ)$ are two isostrophic loops, then:
1. $Z(TMlt(Q,\cdot)) \cong Z(TMlt(Q,\circ))$ (as $TMlt(Q,\cdot) \cong TMlt(Q,\circ)$);
2. $Z(TMlt(Q,\cdot))e_1 \cong Z(TMlt(Q,\circ))e_2$, where e_1 (resp. e_2) is the unit of $(Q,\cdot)$ (resp, $(Q,\circ)$).

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$\{L_x|x\in Z(Q), x^2=e\}$, where $\mathcal{F}'_{(\cdot)}=\{\varphi\in\mathcal{F}_{(\cdot)}|\varphi^*=\varphi^{-1}\}\le\mathcal{F}_{(\cdot)}$;
2. $Z(TMlt(Q))e\trianglelefteq Q$;
3. $Z(TMlt(Q))e=\{a\in Z(Q)|a^2=e\}=$
$\{a\in Q|\varphi(a)=a, \forall\varphi\in TInn(Q,\cdot)\}$.

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Theorem 5 (Statements 3 and 4 are corollaries from Proposition 4)

Let $(Q, \cdot)$ be a commutative Moufang loop, $Z_1 = Z(TMlt(Q, \cdot))e$ be the "small center" of Q and $Z_1^* = \{\alpha \in TMlt(Q) \mid Z_1\alpha(x) = Z_1x, \forall x \in Q\}$.
The following statements hold:
1. $TInn(Q) \leq Aut(Q)$;
2. $Z_1^* \trianglelefteq TMlt(Q)$;
3. $TMlt(Q/Z_1) \cong TMlt(Q)/Z_1^*$;
4. $TInn(Q/Z_1) \cong TInn(Q)/(Z_1^* \cap TInn(Q))$ and $Z_1^*(Q) \cap TInn(Q) \subseteq Z(TInn(Q))$.

Theorem 6

Let Q be a finite loop. The following conditions are equivalent:
(i) TMlt(Q) is a 2-group;
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Middle Bol loops are principal isostrophes of right (left) Bol loops

A loop $(Q, \cdot)$ is called a middle Bol loop if the corresponding e-loop $(Q, \cdot, /, \backslash)$ satisfies the identity $x(yz \setminus x) = (x/z)(y \setminus x)$. A loop $(Q, \circ)$ is middle Bol iff there exists a right (resp. left) Bol loop $(Q, \cdot)$ such that

$$x \circ y = (y \cdot xy^{-1})y = y^{-1} \backslash x \tag{1}$$

$$\textit{resp.} \;\; x \circ y = y(y^{-1}x \cdot y) = x/y^{-1} \tag{2}$$

Corollary from Theorem 1

If $(Q, \cdot)$ is a middle Bol loop and $(Q, \circ)$ is the corresponding left or right Bol loop, then $TMlt(Q, \cdot) = TMlt(Q, \circ)$ and $TInn(Q, \cdot) = TInn(Q, \circ)$.

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Total multiplication groups of middle Bol loops

Theorem 7 (P.Syrbu, I.Grecu)

If $(Q,\cdot)$ is a middle Bol loop, then $Mlt(Q,\cdot) \trianglelefteq TMlt(Q,\cdot)$ and $Inn(Q) \trianglelefteq TInn(Q)$.

Sketch of the proof

$D_x^{(\cdot)} R_z^{(\cdot)} = L_x^{(\cdot)-1} L_{x/z}^{(\cdot)} D_x^{(\cdot)}$, $D_x^{(\cdot)} L_y^{(\cdot)} = L_x^{(\cdot)-1} R_{y\backslash x}^{(\cdot)} R_x^{(\cdot)-1} L_x^{(\cdot)} D_x^{(\cdot)}$
hence $\sigma\varphi\sigma^{-1} \in Mlt(Q)$, for $\forall \sigma \in TMlt(Q), \forall \varphi \in Mlt(Q)$.

Theorem 8 (P.Syrbu, I.Grecu)

If $(Q,\cdot)$ is a middle Bol loop then

$$TMlt(Q)/Mlt(Q) \cong A/(A \cap Mlt(Q)),$$

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Theorem 9 (P.Syrbu, I.Grecu)

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Example 1

Let consider the middle Bol loop $(Q,\cdot)$, where $Q=\{1,2,...,16\}$ and the operation $(\cdot)$ is given by the table:

Example

$(\circ)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	11	2	10	1	14	13	16	15	3	4	5	7	6	9	8
3	3	12	11	1	10	13	14	15	16	2	5	4	6	7	8	9
4	4	10	1	12	11	15	16	14	13	5	3	2	9	8	6	7
5	5	1	10	11	12	16	15	13	14	4	2	3	8	9	7	6
6	6	14	13	16	15	12	11	1	10	7	8	9	3	2	5	4
7	7	13	14	15	16	11	12	10	1	6	9	8	2	3	4	5
8	8	15	16	14	13	1	10	11	12	9	7	6	5	4	2	3
9	9	16	15	13	14	10	1	12	11	8	6	7	4	5	3	2
10	10	3	2	5	4	7	6	9	8	1	12	11	14	13	16	15
11	11	4	5	3	2	8	9	7	6	12	10	1	16	15	13	14
12	12	5	4	2	3	9	8	6	7	11	1	10	15	16	14	13
13	13	7	6	9	8	3	2	5	4	14	15	16	10	1	12	11
14	14	6	7	8	9	2	3	4	5	13	16	15	1	10	11	12
15	15	8	9	7	6	4	5	3	2	16	14	13	11	12	10	1
16	16	9	8	6	7	5	4	2	3	15	13	14	12	11	1	10

.

If Q is a middle Bol loop, then the index of $Mlt(Q)$ in $TMlt(Q)$ may be grater than two

Using GAP System for Computational Discrete Algebra ($https://www.gap-system.org/$), it was found that

$$|Mlt(Q)| = 4096$$

and

$$|TMlt(Q)| = 16384,$$

so $|TMlt(Q) : Mlt(Q)| = 4$.

The following example shows that $Mlt(Q)$ of a right Bol loop Q is not always a normal subgroup of $TMlt(Q)$.

Example 2: A right Bol loop Q such that $Mlt(Q)$ is not a normal subgroup of $TMlt(Q)$

The loop $(Q, \cdot)$, where $Q = \{1, 2, ..., 16\}$ and the operation $(\cdot)$ is given by the table:

$(\cdot)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	11	12	10	1	15	16	13	14	3	4	5	8	9	6	7
3	3	12	11	1	10	16	15	14	13	2	5	4	9	8	7	6
4	4	10	1	12	11	13	14	16	15	5	3	2	6	7	9	8
5	5	1	10	11	12	14	13	15	16	4	2	3	7	6	8	9
6	6	16	15	14	13	12	11	1	10	7	8	9	5	4	3	2
7	7	15	16	13	14	11	12	10	1	6	9	8	4	5	2	3
8	8	14	13	15	16	1	10	11	12	9	7	6	3	2	4	5
9	9	13	14	16	15	10	1	12	11	8	6	7	2	3	5	4
10	10	3	2	5	4	7	6	9	8	1	12	11	14	13	16	15
11	11	4	5	3	2	8	9	7	6	12	10	1	15	16	14	13
12	12	5	4	2	3	9	8	6	7	11	1	10	16	15	13	14
13	13	6	7	8	9	2	3	4	5	14	16	15	10	1	11	12
14	14	7	6	9	8	3	2	5	4	13	15	16	1	10	12	11
15	15	9	8	6	7	5	4	2	3	16	13	14	12	11	10	1
16	16	8	9	7	6	4	5	3	2	15	14	13	11	12	1	10

Q from Example 2 is a right Bol loop with

$$|Mlt(Q)| = 2048$$

and

$$|TMlt(Q)| = 16384,$$

so

$$|TMlt(Q) : Mlt(Q)| = 8.$$

Moreover,
$\varphi = (2,3)(4,5)(6,7)(8,9)(11,12)(13,15,14,16) \in TMlt(Q)$,
$\alpha = (1,2)(3,10)(4,11)(5,12)(6,13,7,14)(8,16,9,15) \in Mlt(Q)$, but
$\varphi\alpha\varphi^{-1} = (13)(2,10)(4,11)(5,12)(6,15,7,16)(8,13,9,14)$
does not belong to $Mlt(Q)$, hence

$$Mlt(Q) \ntrianglelefteq TMlt(Q).$$

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