ON TOTAL MULTIPLICATION GROUPS OF LOOPS

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Multiplication groups of loops are a standard tool of algebraic loop theory. It does not seem likely that the total multiplication groups will achieve a similar status .... Nevertheless, there are situations when middle translations cannot be avoided, for example, when considering the connection to paratopic (isostrophic) loops.

The purpose of this work is to consider some basic facts on the total multiplication groups, by analogy with the known ones on the usual multiplication groups, in particular, to consider the total multiplication groups of some known classes of loops, such as IP-loops, Moufang loops and Bol loops.
It is well known that the notions *multiplication group* and *inner mapping group* of a quasigroup (loop) were introduced by Albert in [Quasigroups I, 1944] and, respectively, by Bruck in [A survey of binary systems, 1958].

Both, the multiplication group and the inner mapping group are invariant (up to isomorphism) under the isotopy of loops.

Later, in 1969 [4], Belousov considered the group generated by all left, right and middle translations of a quasigroup, which he called the "complete multiplication groups", and remarked that such groups are invariant under the parastrophy of quasigroups and play a similar role in defining normal subquasigroups.
Let \((Q, \cdot)\) be a quasigroup. Denote:

\[
Mlt(Q, \cdot) = \langle L_x, R_x \mid x \in Q \rangle - \text{the multiplication group,}
\]
\[
TMlt(Q, \cdot) = \langle L_x, R_x, D_x \mid x \in Q \rangle - \text{the total multiplication group,}
\]
where \(L_x(y) = x \cdot y, \ R_x(y) = y \cdot x, \ D_x(y) = y \setminus x, \forall x, y \in Q.\)

If \(Q\) is a loop with the unit \(e\), then

\[
Inn(Q) = (Mlt(Q))_e - \text{the stabilizer of } e; \quad TInn(Q) = (TMlt(Q))_e - \text{the total inner mapping group of } Q.
\]

The case of \((RIP, LIP)IP−\text{loops (}D_x = R_x I, \text{where } I(x) = x^{-1})\)

**Proposition 1.** If \(Q\) is an \(IP−\text{loop (a group)}\) then:
(i) \(Mlt(Q)\) is a normal subgroup of index two of the group \(TMlt(Q)\);
(ii) \(TMlt(Q) = Mlt(Q) \times < I >, \text{where } I : Q \mapsto Q, I(x) = x^{-1};\)

**Proposition 2.** If \(Q\) is an \(RIP−\text{loop or an } LIP−\text{loop then}\)
\(TMlt(G) = \langle Mlt(Q), I > .\)
Let \((Q, \cdot)\) be a quasigroup. Denote:

\[\text{Mlt}(Q, \cdot) = \langle L_x, R_x | x \in Q \rangle\] - the multiplication group,

\[\text{TMlt}(Q, \cdot) = \langle L_x, R_x, D_x | x \in Q \rangle\] - the total multiplication group,

where \(L_x(y) = x \cdot y\), \(R_x(y) = y \cdot x\), \(D_x(y) = y \setminus x\), \(\forall x, y \in Q\).

If \(Q\) is a loop with the unit \(e\), then

\[\text{Inn}(Q) = (\text{Mlt}(Q))_e\] (the stabilizer of \(e\)) - the inner mapping group of \(Q\);

\[\text{TInn}(Q) = (\text{TMlt}(Q))_e\] - the total inner mapping group of \(Q\).

The case of \((RIP, LIP)IP–\) loops \((D_x = R_x I, \text{where } I(x) = x^{-1})\)

Proposition 1. If \(Q\) is an \(IP–\) loop (a group) then:
(i) \(\text{Mlt}(Q)\) is a normal subgroup of index two of the group \(\text{TMlt}(Q)\);
(ii) \(\text{TMlt}(Q) = \text{Mlt}(Q) \times < I >\), where \(I: Q \mapsto Q, I(x) = x^{-1}\); 

Proposition 2. If \(Q\) is an \(RIP–\) loop or an \(LIP–\) loop then
\(\text{TMlt}(G) = \langle \text{Mlt}(Q), I \rangle\).
Let \((Q, \cdot)\) be a quasigroup. Denote:

\[
Mlt(Q, \cdot) = \langle L_x, R_x \mid x \in Q \rangle \quad \text{- the multiplication group,}
\]

\[
TMlt(Q, \cdot) = \langle L_x, R_x, D_x \mid x \in Q \rangle \quad \text{- the total multiplication group,}
\]

where \(L_x(y) = x \cdot y\), \(R_x(y) = y \cdot x\), \(D_x(y) = y \setminus x\), \(\forall x, y \in Q\).

If \(Q\) is a loop with the unit \(e\), then

\[
Inn(Q) = (Mlt(Q))_e \quad \text{(the stabilizer of \(e\)) - the inner mapping group of \(Q\);} \]

\[
TInn(Q) = (TMlt(Q))_e \quad \text{- the total inner mapping group of \(Q\).}
\]

The case of \((RIP, LIP)\) loops \((D_x = R_x I, \text{where } I(x) = x^{-1})\)

**Proposition 1.** If \(Q\) is an \(IP\)– loop (a group) then:

(i) \(Mlt(Q)\) is a normal subgroup of index two of the group \(TMlt(Q)\);

(ii) \(TMlt(Q) = Mlt(Q) \times \langle I \rangle\), where \(I : Q \mapsto Q, I(x) = x^{-1}\);

**Proposition 2.** If \(Q\) is an \(RIP\)– loop or an \(LIP\)– loop then

\(TMlt(G) = \langle Mlt(Q), I \rangle\).
The behavior of the total multiplication groups under the isostrophy of loops

**Definition.** Two quasigroups are called isostrophic (principally isostrophic) if they have parastrophes that are isotopic (principally isotopic).

Using the fact that every quasigroup operation has six parastrophes (some of them may coincide) we get:

**Theorem 1**

If \((Q, \cdot)\) and \((Q, \circ)\) are isostrophic (principally isostrophic) loops than:

1. \(TMLt(Q, \cdot) \cong TMLt(Q, \circ)\) (resp. \(TMLt(Q, \cdot) = TMLt(Q, \circ)\));
2. \(Tlnn(Q, \cdot) \cong Tlnn(Q, \circ)\) (resp. \(Tlnn(Q, \cdot) = Tlnn(Q, \circ)\)).

**Remark**

In a general case, usual multiplication groups are not invariant (up to isomorphism) under the isostrophy of loops.
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(ii) \(TInn(Q, \cdot) \cong TInn(Q, \circ)\) (resp. \(TInn(Q, \cdot) = TInn(Q, \circ)\)).

**Remark**

In a general case, usual multiplication groups are not invariant (up to isomorphism) under the isostrophy of loops.
1. $\text{Inn}_h(Q) = \langle L_{x,y}, R_{x,y}, T_x | x, y \in Q \rangle$, where 

$$L_{x,y} = L_{x \cdot y}^{-1} L_x L_y, R_{x,y} = R_{x \cdot y}^{-1} R_y R_x, T_x = L_x^{-1} R_x,$$

$$x \circ y = R_h^{-1}(a \cdot R_h b), x \star y = L_h^{-1}(L_h \cdot b), \sigma = R_h^{-1} L_h;$$

2. If $(Q, \cdot)$ is a loop with the unit $e$, then the generators of $\text{Inn}_e(Q) = \text{Inn}(Q)$ are:

$$L_{x,y} = L_{x \cdot y}^{-1} L_x L_y, R_{x,y} = R_{x \cdot y}^{-1} R_y R_x, T_x = L_x^{-1} R_x.$$

3. Let $(Q, \cdot)$ be a quasigroup. Belousov proved in [4] that the group $T\text{Inn}_h(Q)$ is generated by the set of mappings

$$L^{(\cdot)}_{x,y}, R^{(\cdot)}_{x,y}, T^{(\cdot)}_x, N^{(\cdot)}_{x,y}, S^{(\cdot)}_x,$$

where $N^{(\cdot)}_{x,y} = L^{(\cdot)}_{x,y}$, $S^{(\cdot)}_x = T^{(\cdot)}_x$, for $\forall x, y \in Q$.

In fact, at present there are known several sets of generators of the total inner mapping group.
A set of generators of the total inner mapping group of a loop

**Theorem 2**
If \((Q, \cdot)\) is a loop, then
\[
TInn(Q) = \langle L_{x,y}, R_{x,y}, T_x, P_{x,y}, U_x | x, y \in Q \rangle,
\]
where
\[
L_{x,y} = L_x^{-1} L_x L_y, \quad R_{x,y} = R_{x,y}^{-1} R_y R_x, \quad T_x = L_x^{-1} R_x,
\]
\[
P_{x,y} = R_{y}^{-1} L_x D_y D_x, \quad U_x = D_x R_x.
\]

**Corollary 1**
If \((Q, \cdot)\) is a power associative loop, then
\[
TInn(Q) = \langle L_{x,y}, R_{x,y}, P_{x,y}, U_x | x, y \in Q \rangle,
\]
where
\[
L_{x,y} = L_x^{-1} L_x L_y, \quad R_{x,y} = R_{x,y}^{-1} R_y R_x, \quad P_{x,y} = R_y^{-1} L_x D_y D_x, \quad U_x = D_x R_x.
\]

**Corollary 2**
If \((Q, \cdot)\) is a middle Bol loop, then
\[
TInn(Q) = \langle R_{x,y}, P_{x,y}, U_x | x, y \in Q \rangle.
\]
A set of generators of the total inner mapping group of a loop

**Theorem 2**

If \((Q, \cdot)\) is a loop, then

\[ TInn(Q) = \langle L_{x,y}, R_{x,y}, T_x, P_{x,y}, U_x \mid x, y \in Q \rangle, \]

where

\[ L_{x,y} = L_x^{-1} L_x L_y, \quad R_{x,y} = R_y^{-1} R_y R_x, \quad T_x = L_x^{-1} R_x, \]
\[ P_{x,y} = R_y^{-1} L_x D_y D_x, \quad U_x = D_x R_x. \]

**Corollary 1**

If \((Q, \cdot)\) is a power associative loop, then

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\[ U_x = D_x R_x. \]

**Corollary 2**

If \((Q, \cdot)\) is a middle Bol loop, then

\[ TInn(Q) = \langle R_{x,y}, P_{x,y}, U_x \mid x, y \in Q \rangle. \]
A set of generators of the total inner mapping group of a loop

Theorem 2

If \((Q, \cdot)\) is a loop, then
\[
TInn(Q) = \langle L_{x,y}, R_{x,y}, T_{x}, P_{x,y}, U_{x} | x, y \in Q \rangle,
\]
where
\[
L_{x,y} = L_{x \cdot y}^{-1}L_{x}L_{y}, \quad R_{x,y} = R_{x \cdot y}^{-1}R_{y}R_{x}, \quad T_{x} = L_{x}^{-1}R_{x},
\]
\[
P_{x,y} = R_{y}^{-1}L_{x}D_{y}D_{x}, \quad U_{x} = D_{x}R_{x}.
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\]
\[
U_{x} = D_{x}R_{x}.
\]

Corollary 2

If \((Q, \cdot)\) is a middle Bol loop, then
\[
TInn(Q) = \langle R_{x,y}, P_{x,y}, U_{x} | x, y \in Q \rangle.
\]
Analog properties to that of $Mlt(Q)$ and $Inn(Q)$

**Proposition 3.** ($\xi : x \mapsto L_x TInn(Q)$ is a bijection)

Let $(Q, \cdot)$ be a quasigroup and $h \in Q$. The following statements hold:
1. If $TInn_h(Q) = \{\varepsilon\}$ then $(Q, \cdot)$ is an abelian group;
2. If $(Q, \cdot)$ is a finite quasigroup, then
   $$|TInn_h(Q)| = |Q||TInn_h(Q)|.$$

**Proposition 4.** ($\varphi : P_{x_1}^{\varepsilon_1} \ldots P_{x_n}^{\varepsilon_n} \mapsto P_{\theta x_1}^{\varepsilon_1} \ldots P_{\theta x_n}^{\varepsilon_n}$)

Let $\theta : Q \mapsto G$ be a surjective homomomorphism of loops, from $(Q, \cdot)$ to $(G, \circ)$. Then $\theta$ induces a homomorphism $\varphi : TMLt(Q) \mapsto TMLt(G)$, such that $\varphi(\alpha)\theta = \theta\alpha, \forall \alpha \in TMLt(Q)$, and the following statements hold:
1. $TMLt(Q/H) \cong TMLt(Q)/H^*$;
2. $TInn(Q/H) \cong TInn(Q)/(H^* \cap TInn(Q))$,
where $H^* = \{\alpha \in TMLt(Q) | H(\alpha(x)) = Hx, \forall x \in Q\}$, $H = \text{Ker}\varphi$. 

P. Syrbu

On total multiplication groups of loops
Analog properties to that of \( \text{Mlt}(Q) \) and \( \text{Inn}(Q) \)

**Proposition 3.** \((\xi : x \mapsto L_x TInn(Q) \text{ is a bijection)}\)

Let \((Q, \cdot)\) be a quasigroup and \(h \in Q\). The following statements hold:

1. If \(TInn_h(Q) = \{\varepsilon\}\) then \((Q, \cdot)\) is an abelian group;
2. If \((Q, \cdot)\) is a finite quasigroup, then
   \[
   |T\text{Mlt}(Q)| = |Q||T\text{Inn}_h(Q)|.
   \]

**Proposition 4.** \((\varphi : P_{x_1}^{\varepsilon_1} \cdots P_{x_n}^{\varepsilon_n} \mapsto P_{\theta x_1}^{\varepsilon_1} \cdots P_{\theta x_n}^{\varepsilon_n} )\)

Let \(\theta : Q \mapsto G\) be a surjective homomorphism of loops, from \((Q, \cdot)\) to \((G, \circ)\). Then \(\theta\) induces a homomorphism \(\varphi : T\text{Mlt}(Q) \mapsto T\text{Mlt}(G)\), such that \(\varphi(\alpha) \theta = \theta \alpha, \forall \alpha \in T\text{Mlt}(Q)\), and the following statements hold:

1. \(T\text{Mlt}(Q/H) \cong T\text{Mlt}(Q)/H^*\);
2. \(T\text{Inn}(Q/H) \cong T\text{Inn}(Q)/(H^* \cap T\text{Inn}(Q))\),
where \(H^* = \{\alpha \in T\text{Mlt}(Q) | H(\alpha(x)) = Hx, \forall x \in Q\}\), \(H = \text{Ker}\varphi\).
Let \((Q, \cdot)\) be a loop. The nucleus, commutant and the centre of \((Q, \cdot)\) are defined, respectively, as follows:
\[
N = \{ a \in Q \mid ax \cdot y = a \cdot xy, xa \cdot y = x \cdot ay, xy \cdot a = x \cdot ya, \forall x, y \in Q \}
\]
\[
C = \{ a \in Q \mid ax = xa, \forall x \in Q \}
\]
and \(Z(Q) = N \cap C\).

**Theorem 3**

If \((Q, \cdot)\) is a loop with the unit \(e\), then
1. \(Z(TMlt(Q, \cdot)) = \mathcal{L}(\cdot) \cap R(\cdot) \cap \mathcal{F}'(\cdot) = \{ \varphi \in Z(Mlt(Q, \cdot)) \mid \varphi^2 = e \} = \{ L_x \mid x \in Z(Q), x^2 = e \}\), where \(\mathcal{F}'(\cdot) = \{ \varphi \in \mathcal{F}(\cdot) \mid \varphi^* = \varphi^{-1} \} \leq \mathcal{F}(\cdot)\);
2. \(Z(TMlt(Q))e \trianglelefteq Q\);
3. \(Z(TMlt(Q))e = \{ a \in Z(Q) \mid a^2 = e \} = \{ a \in Q \mid \varphi(a) = a, \forall \varphi \in TInn(Q, \cdot) \}\).

**Theorem 4**

If \((Q, \cdot)\) and \((Q, \circ)\) are two isostrophic loops, then:
1. \(Z(TMlt(Q, \cdot)) \cong Z(TMlt(Q, \circ))\) (as \(TMlt(Q, \cdot) \cong TMlt(Q, \circ)\));
2. \(Z(TMlt(Q, \cdot))e_1 \cong Z(TMlt(Q, \circ))e_2\), where \(e_1\) (resp. \(e_2\)) is the unit of \((Q, \cdot)\) (resp. \((Q, \circ)\)).
The center of the total multiplication group

Let \((Q, \cdot)\) be a loop. The *nucleus*, *commutant* and the *centre* of \((Q, \cdot)\) are defined, respectively, as follows:

\[ N = \{ a \in Q | ax \cdot y = a \cdot xy, xa \cdot y = x \cdot ay, xy \cdot a = x \cdot ya, \forall x, y \in Q \} \]

\[ C = \{ a \in Q | ax = xa, \forall x \in Q \} \]

and \( Z(Q) = N \cap C \).

**Theorem 3**

If \((Q, \cdot)\) is a loop with the unit \(e\), then

1. \( Z(TMlt(Q, \cdot)) = \mathcal{L}(\cdot) \cap \mathcal{R}(\cdot) \cap \mathcal{F}'(\cdot) = \{ \varphi \in Z(Mlt(Q, \cdot)) | \varphi^2 = e \} = \{ L_x | x \in Z(Q), x^2 = e \} \), where \( \mathcal{F}'(\cdot) = \{ \varphi \in \mathcal{F}(\cdot) | \varphi^* = \varphi^{-1} \} \leq \mathcal{F}(\cdot) \);
2. \( Z(TMlt(Q))e \trianglelefteq Q \);
3. \( Z(TMlt(Q))e = \{ a \in Z(Q) | a^2 = e \} = \{ a \in Q | \varphi(a) = a, \forall \varphi \in TInn(Q, \cdot) \} \).

**Theorem 4**

If \((Q, \cdot)\) and \((Q, \circ)\) are two isostrophic loops, then:

1. \( Z(TMlt(Q, \cdot)) \cong Z(TMlt(Q, \circ)) \) (as \( TMlt(Q, \cdot) \cong TMlt(Q, \circ) \));
2. \( Z(TMlt(Q, \cdot))e_1 \cong Z(TMlt(Q, \circ))e_2 \), where \( e_1 \) (resp. \( e_2 \)) is the unit of \((Q, \cdot)\) (resp. \((Q, \circ)\)).
Theorem 5 (Statements 3 and 4 are corollaries from Proposition 4)

Let \((Q, \cdot)\) be a commutative Moufang loop, \(Z_1 = Z(TMlt(Q, \cdot))e\) be the "small center" of \(Q\) and \(Z_1^* = \{\alpha \in TMlt(Q) \mid Z_1 \alpha(x) = Z_1 x, \forall x \in Q\}\).

The following statements hold:
1. \(Tlnn(Q) \leq Aut(Q)\);
2. \(Z_1^* \trianglelefteq TMlt(Q)\);
3. \(TMlt(Q/Z_1) \cong TMlt(Q)/Z_1^*\);
4. \(Tlnn(Q/Z_1) \cong Tlnn(Q)/(Z_1^* \cap Tlnn(Q))\) and \(Z_1^*(Q) \cap Tlnn(Q) \subseteq Z(Tlnn(Q))\).

Theorem 6

Let \(Q\) be a finite loop. The following conditions are equivalent:
(i) \(TMlt(Q)\) is a 2-group;
(ii) \(TMlt(Q)\) is nilpotent;
(iii) \(Q\) is a nilpotent loop of order \(2^k, k \geq 0\).
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1. $TInn(Q) \leq Aut(Q)$;
2. $Z_1^* \triangleleft TMlt(Q)$;
3. $TMlt(Q/Z_1) \cong TMlt(Q)/Z_1^*$;
4. $TInn(Q/Z_1) \cong TInn(Q)/(Z_1^* \cap TInn(Q))$ and $Z_1^*(Q) \cap TInn(Q) \subseteq Z(TInn(Q))$.

Theorem 6

Let $Q$ be a finite loop. The following conditions are equivalent:

(i) $TMlt(Q)$ is a 2-group;
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(iii) $Q$ is a nilpotent loop of order $2^k$, $k \geq 0$. 

P. Syrbu
On total multiplication groups of loops
A loop $(Q, \cdot)$ is called a middle Bol loop if the corresponding e-loop $(Q, \cdot, /, \backslash)$ satisfies the identity $x(yz \backslash x) = (x/z)(y \backslash x)$. A loop $(Q, \circ)$ is middle Bol iff there exists a right (resp. left) Bol loop $(Q, \cdot)$ such that

$$x \circ y = (y \cdot xy^{-1})y = y^{-1} \backslash x$$  \hspace{1cm} (1)

resp. $x \circ y = y(y^{-1}x \cdot y) = x/y^{-1}$  \hspace{1cm} (2)

Corollary from Theorem 1

If $(Q, \cdot)$ is a middle Bol loop and $(Q, \circ)$ is the corresponding left or right Bol loop, then $TMlt(Q, \cdot) = TMlt(Q, \circ)$ and $TInn(Q, \cdot) = TInn(Q, \circ)$.
A loop $(Q, \cdot)$ is called a middle Bol loop if the corresponding e-loop $(Q, \cdot, /, \backslash)$ satisfies the identity $x(yz \backslash x) = (x/z)(y \backslash x)$. A loop $(Q, \circ)$ is middle Bol iff there exists a right (resp. left) Bol loop $(Q, \cdot)$ such that

$$x \circ y = (y \cdot xy^{-1})y = y^{-1} \backslash x$$  \hspace{1cm} (1)

resp.

$$x \circ y = y(y^{-1} x \cdot y) = x/y^{-1}$$  \hspace{1cm} (2)

Corollary from Theorem 1

If $(Q, \cdot)$ is a middle Bol loop and $(Q, \circ)$ is the corresponding left or right Bol loop, then $TMlt(Q, \cdot) = TMlt(Q, \circ)$ and $TInn(Q, \cdot) = TInn(Q, \circ)$.
Theorem 7 (P. Syrbu, I. Grecu)

If \((Q, \cdot)\) is a middle Bol loop, then \(\text{Mlt}(Q, \cdot) \leq \text{TMLt}(Q, \cdot)\) and \(\text{Inn}(Q) \leq \text{TInn}(Q)\).

Sketch of the proof

\[
D_x^{(\cdot)} R_z^{(\cdot)} = L_x^{(\cdot)^{-1}} L_{x/z}^{(\cdot)} D_x^{(\cdot)} ,\quad D_x^{(\cdot)} L_y^{(\cdot)} = L_x^{(\cdot)^{-1}} R_y^{(\cdot)} R_x^{(\cdot)^{-1}} L_x^{(\cdot)} D_x^{(\cdot)}
\]

hence \(\sigma \varphi \sigma^{-1} \in \text{Mlt}(Q)\), for \(\forall \sigma \in \text{TMLt}(Q), \forall \varphi \in \text{Mlt}(Q)\).

Theorem 8 (P. Syrbu, I. Grecu)

If \((Q, \cdot)\) is a middle Bol loop then

\[\text{TMLt}(Q)/\text{Mlt}(Q) \cong A/(A \cap \text{Mlt}(Q)),\]

where \(A = \langle D_x \mid x \in Q \rangle\).

Theorem 9 (P. Syrbu, I. Grecu)

If \((Q, \cdot)\) is a middle Bol loop then

\[\text{TInn}(Q)/\text{Inn}(Q) \cong B/(B \cap \text{Inn}(Q)),\]

where \(B = \langle P_{x,y}, U_x \mid x, y \in Q \rangle\).
Theorem 7 (P. Syrbu, I. Grecu)

If \((Q, \cdot)\) is a middle Bol loop, then \(\text{Mlt}(Q, \cdot) \leq T\text{Mlt}(Q, \cdot)\) and \(\text{Inn}(Q) \leq T\text{Inn}(Q)\).

Sketch of the proof

\[
D_x^\cdot R_z^\cdot = L_x^{\cdot -1} L_z^{\cdot} D_x^{\cdot}, \quad D_x^\cdot L_y^\cdot = L_x^{\cdot -1} R_y^{\cdot} R_x^{\cdot -1} L_x^{\cdot} D_x^{\cdot}
\]

hence \(\sigma \varphi \sigma^{-1} \in \text{Mlt}(Q)\), for \(\forall \sigma \in T\text{Mlt}(Q), \forall \varphi \in \text{Mlt}(Q)\).

Theorem 8 (P. Syrbu, I. Grecu)

If \((Q, \cdot)\) is a middle Bol loop then

\[T\text{Mlt}(Q)/\text{Mlt}(Q) \cong A/(A \cap \text{Mlt}(Q)),\]

where \(A = \langle D_x \mid x \in Q \rangle\).

Theorem 9 (P. Syrbu, I. Grecu)

If \((Q, \cdot)\) is a middle Bol loop then

\[T\text{Inn}(Q)/\text{Inn}(Q) \cong B/(B \cap \text{Inn}(Q)),\]

where \(B = \langle P_{x,y}; U_x \mid x, y \in Q \rangle\).
## Total multiplication groups of middle Bol loops

### Theorem 7 (P. Syrbu, I. Grecu)

If \((Q, \cdot)\) is a middle Bol loop, then \(Mlt(Q, \cdot) \leq T\!Mlt(Q, \cdot)\) and \(Inn(Q) \leq T\!Inn(Q)\).

### Sketch of the proof

\[
D_x^{(\cdot)} R_z^{(\cdot)} = L_x^{(\cdot)} L_x^{(\cdot)} D_x^{(\cdot)}, \quad D_x^{(\cdot)} L_y^{(\cdot)} = L_x^{(\cdot)} R_y^{(\cdot)} R_x^{(\cdot)} L_x^{(\cdot)} D_x^{(\cdot)}
\]

hence \(\sigma \varphi \sigma^{-1} \in Mlt(Q)\), for \(\forall \sigma \in T\!Mlt(Q), \forall \varphi \in Mlt(Q)\).

### Theorem 8 (P. Syrbu, I. Grecu)

If \((Q, \cdot)\) is a middle Bol loop then
\[
T\!Mlt(Q)/Mlt(Q) \cong A/(A \cap Mlt(Q)),
\]
where \(A = \langle D_x \mid x \in Q \rangle\).

### Theorem 9 (P. Syrbu, I. Grecu)

If \((Q, \cdot)\) is a middle Bol loop then
\[
T\!Inn(Q)/Inn(Q) \cong B/(B \cap Inn(Q)),
\]
where \(B = \langle P_{x,y}; U_x \mid x, y \in Q \rangle\).
Total multiplication groups of middle Bol loops

**Theorem 7 (P. Syrbu, I. Grecu)**

If \((Q, \cdot)\) is a middle Bol loop, then \(\text{Mlt}(Q, \cdot) \trianglelefteq \text{TMlt}(Q, \cdot)\) and \(\text{Inn}(Q) \trianglelefteq \text{TInn}(Q)\).

**Sketch of the proof**

\[ D_x^{(\cdot)} R_z^{(\cdot)} = L_x^{(\cdot)-1} L_{x/z}^{(\cdot)} D_x^{(\cdot)}, \quad D_x^{(\cdot)} L_y^{(\cdot)} = L_x^{(\cdot)-1} R_y^{(\cdot)-1} R_x^{(\cdot)-1} L_x^{(\cdot)} D_x^{(\cdot)} \]

hence \(\sigma \varphi \sigma^{-1} \in \text{Mlt}(Q)\), for \(\forall \sigma \in \text{TMlt}(Q), \forall \varphi \in \text{Mlt}(Q)\).

**Theorem 8 (P. Syrbu, I. Grecu)**

If \((Q, \cdot)\) is a middle Bol loop then

\[ \text{TMlt}(Q) / \text{Mlt}(Q) \cong A / (A \cap \text{Mlt}(Q)), \]

where \(A = \langle D_x \mid x \in Q \rangle\).

**Theorem 9 (P. Syrbu, I. Grecu)**

If \((Q, \cdot)\) is a middle Bol loop then

\[ \text{TInn}(Q) / \text{Inn}(Q) \cong B / (B \cap \text{Inn}(Q)), \]

where \(B = \langle P_{x,y}, U_x \mid x, y \in Q \rangle\).
Let consider the middle Bol loop \((Q, \cdot)\), where \(Q = \{1, 2, \ldots, 16\}\) and the operation \((\cdot)\) is given by the table:

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If $Q$ is a middle Bol loop, then the index of $\text{Mlt}(Q)$ in $\text{TMLt}(Q)$ may be greater than two.

Using GAP System for Computational Discrete Algebra (https://www.gap-system.org/), it was found that

$$|\text{Mlt}(Q)| = 4096$$

and

$$|\text{TMLt}(Q)| = 16384,$$

so $|\text{TMLt}(Q) : \text{Mlt}(Q)| = 4$.

The following example shows that $\text{Mlt}(Q)$ of a right Bol loop $Q$ is not always a normal subgroup of $\text{TMLt}(Q)$. 

P. Syrбу

On total multiplication groups of loops
Example 2: A right Bol loop $Q$ such that $\text{Mlt}(Q)$ is not a normal subgroup of $\text{TMLt}(Q)$

The loop $(Q, \cdot)$, where $Q = \{1, 2, \ldots, 16\}$ and the operation $(\cdot)$ is given by the table:

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$Q$ from Example 2 is a right Bol loop with

$$|\text{Mlt}(Q)| = 2048$$

and

$$|\text{T Mlt}(Q)| = 16384,$$

so

$$|\text{T Mlt}(Q) : \text{Mlt}(Q)| = 8.$$

Moreover,

$\varphi = (2,3)(4,5)(6,7)(8,9)(11,12)(13,15,14,16) \in \text{T Mlt}(Q),$

$\alpha = (1,2)(3,10)(4,11)(5,12)(6,13,7,14)(8,16,9,15) \in \text{Mlt}(Q),$ but

$\varphi \alpha \varphi^{-1} = (13)(2,10)(4,11)(5,12)(6,15,7,16)(8,13,9,14)$

does not belong to $\text{Mlt}(Q),$ hence

$$\text{Mlt}(Q) \not\triangleleft \text{T Mlt}(Q).$$
1. A. A. Albert, Quasigroups I,II. Trans. Amer. Math. Soc. 54 (1943), 507 - 519; 55 (1944), 401 - 419.


3. V. Belousov, Foundations of the theory of quasigroups and loops, Nauka, Moscow, 1967. (Russian)

