Functional method in quasigroup theory

Fedir Sokhatsky

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Algebraic definition

An algebra $(Q; f, {}^{\ell}f, {}^{r}f)$ is called a *quasigroup*, if

$$f({}^{\ell}f(x,y),y) = x, \qquad {}^{\ell}f(f(x,y),y) = x,$$
$$f(x,{}^{r}f(x,y)) = y, \qquad {}^{r}f(x,f(x,y)) = y. \tag{1}$$

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Functional definition

Consider *left multiplication* $\underset{\ell}{\oplus}$ and *right multiplication* $\underset{r}{\oplus}$:

$$(f \underset{\ell}{\oplus} g)(x,y) := f(g(x,y),y), \qquad (f \underset{r}{\oplus} g)(x,y) := f(x,g(x,y)).$$

Both of them are associative and functions ι_1, ι_2, where $\iota_1(x,y) := x$ and $\iota_2(x,y) := y$ are their neutrals:

$$f \underset{\ell}{\oplus} e_1 = f = e_1 \underset{\ell}{\oplus} f, \qquad f \underset{r}{\oplus} e_2 = f = e_2 \underset{r}{\oplus} f.$$

$(\mathcal{O}_2; \underset{\ell}{\oplus}, e_1)$ is a left symmetric semigroups of binary operations.

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Which elements are invertible in this semigroups?

In the left symmetric semigroup: $f \underset{\ell}{\oplus} {}^{\ell}f = e_1$, ${}^{\ell}f \underset{\ell}{\oplus} f = e_1$, i.e.

$$f({}^{\ell}f(x,y),y) = x, \qquad {}^{\ell}f(f(x,y),y) = x.$$

In the right symmetric semigroup: $f \underset{r}{\oplus} {}^{r}f = e_1$, ${}^{r}f \underset{r}{\oplus} f = e_2$, i.e.

$$f(x, {}^{r}f(x,y)) = y, \qquad {}^{r}f(x, f(x,y)) = y.$$

Functional definition of a quasigroup

An algebra $(Q; {}^{\ell}f, {}^{r}f)$ is called a quasigroup, if f is invertible element in both left and right symmetric semigroups, ${}^{\ell}f$ is its inverse in the left symmetric semigroup and ${}^{r}f$ is its inverse in the right symmetric semigroup.

Let Δ_2 denote the set of all invertible operations on the same carrier.

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Let Δ_2 denote the set of all invertible operations on the same carrier.

Let F_i be functional variables taking their values in Δ_2. A functional equation is a universally quantified equality of two terms consisting of functional and individual variables. For example,

$$(\forall x)(\forall y)(\forall z) \qquad F_1(F_2(x,y),z) = F_3(x,F_4(y,z)) \qquad (1)$$

is a functional equation of associativity. Moreover, it is generalized because all functional variables are pairwise different.

Theorem (Belousov,1958)

A quadruble (f_1, f_2, f_3, f_4) of functions from Δ_2 is a solution of (1) iff there is a group $(Q;+,0)$ bijections α, β, γ, δ, ν, such that

$$f_1(u,z) = \delta u + \gamma z, \qquad f_2(y,z) = \delta^{-1}(\alpha x + \beta y),$$
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Application in mathematical analysis

A topological quasigroup (invertible function) on real numbers is a binary function which is monotonic on each of its arguments. For example, $f(x, y) := \sqrt[11]{x^7 + x^{21}}$

Corollary

A quadruble (f_1, f_2, f_3, f_4) of binary real monotonic functions is a solution of the functional equation of general associativity iff there are monotonic α, β, γ, δ, ν, φ such that

$$f_1(u, z) = \varphi(\delta u + \gamma z), \qquad f_2(y, z) = \delta^{-1}(\alpha x + \beta y),$$
$$f_3(x, u) = \varphi(\alpha x + \nu u), \qquad f_4(y, z) = \nu^{-1}(\beta y + \gamma z),$$

where $(+)$ is the addition of real numbers.

An application in algebra: group isotopy property

An identity is said to have a *group isotopy property*, if every quasigroup satisfying this identity is isotopic to a group.

Variables $x_1, \ldots, x_n$ are said to be *isolated in an identity by sub-terms* $t_1, \ldots, t_k$, if all appearances in the identity of each of these variables belong to exactly two of these terms and each variable has only one appearance in at least one of the terms.

Let x, y, z be fixed variables. We will write $t(x, y)$, if the term t contains the variables x and y and does not contain z.

Theorem (F. Sokhatsky, 2019)

A quasigroup identity has a group isotopic property if three of its variables x, y, z are isolated by some sub-terms $t_1(x, y)$, $t_2(x, z)$, $t_3(y, z)$.

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Example of an identities with the group isotopy property

For example, each quasigroup satisfying the identity

$$\Big((u^{n_1}((x\cdot(xu)^{n_2}y)\cdot x^{n_3})\cdot u^{n_4})\cdot(v\cdot(z^{n_5}x\cdot zu)v)u)\Big)\cdot(y\cdot(zu^{n_6})y^{n})) = v$$

is isotopic to a group. Bracketing in u^{n_1}, $(xu)^{n_2}$,... does not matter.

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Parastrophic symmetry

Parastrophes of an invertible operation

Let $(Q; f, {}^{\ell}f, {}^{r}f)$ be a quasigroup. All *parastrophes of f*:

$$ {}^{\sigma}f(x_{1\sigma}, x_{2\sigma}) = x_{3\sigma} :\Leftrightarrow f(x_1, x_2) = x_3, $$

$\sigma \in S_3 := \{\iota, \ell, r, s, s\ell, sr\}$, $\ell := (13)$, $r := (23)$, $s := (12)$.

Parastrophes of a class of quasigroups

Let $\mathfrak{A}$ be a class of quasigroups, ${}^{\sigma}\mathfrak{A}$ be a class of all σ-parastrophes of quasigroups from the class $\mathfrak{A}$. The class ${}^{\sigma}\mathfrak{A}$ is called *σ-parastrophe of $\mathfrak{A}$*.

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Parastrophes of a proposition

σ-parastrophe of a proposition P in a class of quasigroups $\mathfrak{A}$ is called a proposition ${}^{\sigma}P$ obtained from P by replacing the main operation with its σ^{-1}-parastrophe.

Since the relationships

$$^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f, \qquad {}^{\sigma}({}^{\tau}\mathfrak{A}) = {}^{\sigma\tau}\mathfrak{A}, \qquad {}^{\sigma}({}^{\tau}P) = {}^{\sigma\tau}P \tag{1}$$

are true, then S_3 acts

1. on the set Δ_2 of all binary invertible operations defined on the same carrier;
2. on the set of pairwise parastrophic classes of quasigroups;
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Let S_3 acts on a set K

The action is called a *parastrophy action*. The stabilizer under this action is the *group of parastrophic symmetries* $\mathrm{Ps}(k)$. An orbit is a *truss* $\mathrm{Tr}(k)$. The well-known formula

$$|\mathrm{Ps}(k)| \cdot |\mathrm{Tr}(k)| = |S_3| = 3! = 6$$

and therefore the number of different elements in a truss is 1,2,3,6, if $|\mathrm{Ps}(k)| = 6, 3, 2, 1$. An element k will be called:

- *totally symmetric*, if $|\mathrm{Ps}(k)| = 6$, i.e. $\mathrm{Ps}(k) = S_3$;
- *strictly semisymmetric*, if $|\mathrm{Ps}(k)| = 3$, i.e. $\mathrm{Ps}(k) = A_3$;
- *semisymmetric*, if $|\mathrm{Ps}(k)|$ is 3 or 6, i.e. $\mathrm{Ps}(k)$ is A_3 or S_3;
- *strictly one-sided symmetric*, if $|\mathrm{Ps}(k)| = 2$, i.e. $\mathrm{Ps}(k)$ is $\{\iota, s\}$ or $\{\iota, \ell\}$ or $\{\iota, r\}$;
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Examples of varieties

- totally symmetric varieties: 1) all quasigroups; 2) distributive quasigroups; 3) first Shredder quasigroups $xy \cdot y = x \cdot xy$. 4) second Shredder quasigroups $xy \cdot yx = x$; 5) total symmetric quasigroups; 6) semisymmetric quasigroups;
- Semisymmetric varieties: 1) $x(yz \cdot zx) = y$ (has been found by H. Krainichuk);
- One-sided symmetric varieties: 1) second Stein $x \cdot xy = yx$; 2) groups; 3) Moufang loops;
- Asymmetric varieties: 1) first Stein $x \cdot xy = yx$; 2) second Belousov $y(x \cdot xy) = x$.

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Theorem (Sokhatsky)

A proposition P is true in a class of quasigroups $\mathfrak{A}$ iff the proposition ${}^{\sigma}P$ is true in the class ${}^{\sigma}\mathfrak{A}$, for all $\sigma \in S_3$.

Corollary

For all σ from $\mathrm{Ps}(\mathfrak{A})$, the proposition ${}^{\sigma}P$ is true iff the proposition P is true.

Corollary

Let $\mathfrak{A}$ be a total symmetric variety. If P is a true proposition in $\mathfrak{A}$, then ${}^{\sigma}P$ is also true or all $\sigma \in S_3$.

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For all σ from $\mathrm{Ps}(\mathfrak{A})$, the proposition ${}^{\sigma}P$ is true iff the proposition P is true.

Corollary

Let $\mathfrak{A}$ be a total symmetric variety. If P is a true proposition in $\mathfrak{A}$, then ${}^{\sigma}P$ is also true or all $\sigma \in S_3$.

Theorem (Sokhatsky)

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An example

Consider the well-known statement P: "Every quasigroup is isotopic to a loop". The statement "to be a quasigroup" is totally symmetric, the statement "to be isotopic" is also totally symmetric, but the statement "to be a loop" is not. Consider all parastrophes of the statement "to be a left neutral element", one can obtain three notions of a one-sided neutrality:

$$e \cdot x = x, \quad \text{a left neutral element,}$$

$$x \cdot e = x, \quad \text{a right neutral element,}$$

$$x \cdot x = e, \quad \text{a middle neutral element.}$$

As a result, we have three notions of loops:

a left-right loop, a left-middle loop, a right-middle loop.

The left-right loop is usually called a loop. That is why we have three pairwise parastrophic statements:

1. P: "Every quasigroup is isotopic to a loop";
2. ${}^{\ell}P$: "Every quasigroup is isotopic to a right-middle loop";
3. ${}^{r}P$: "Every quasigroup is isotopic to a left-middle loop".

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