Augmented quasigroups:

from group duals to Heyting algebras

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for each object A of $\mathbf{V}$.

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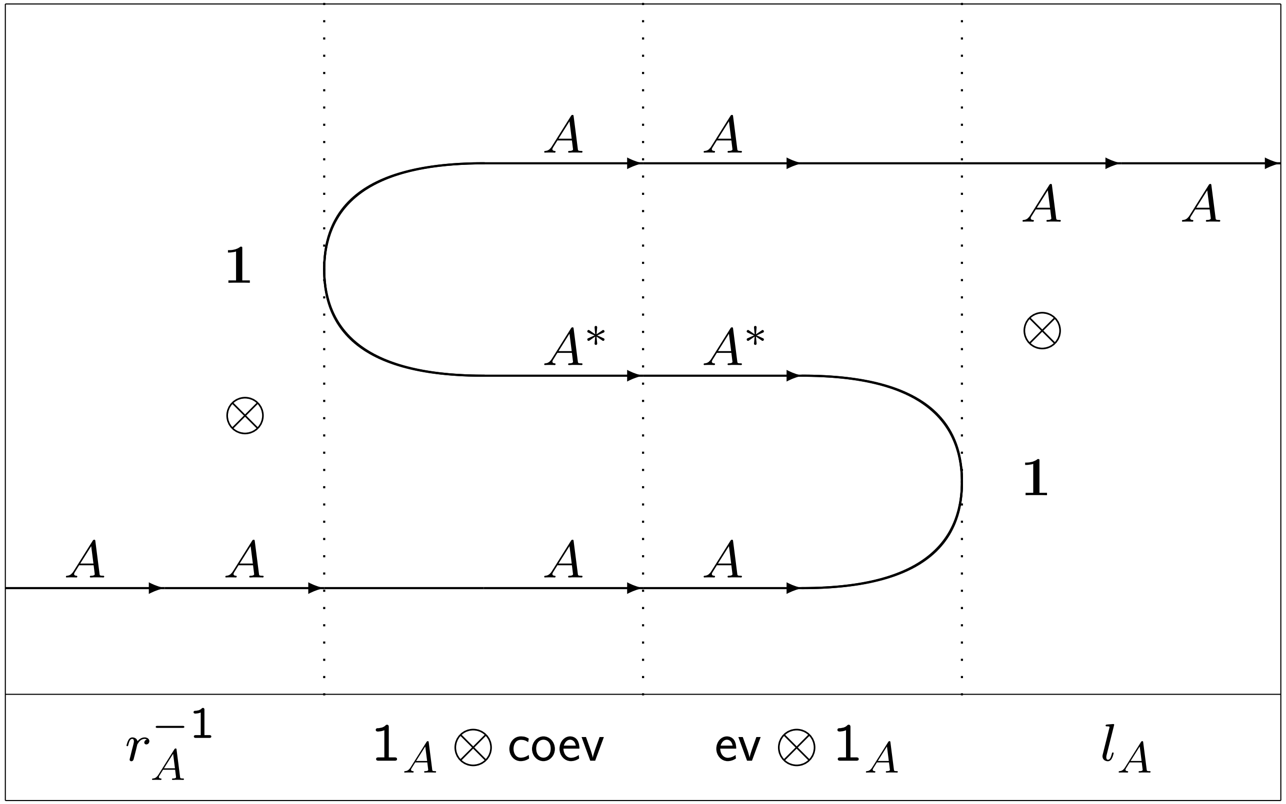

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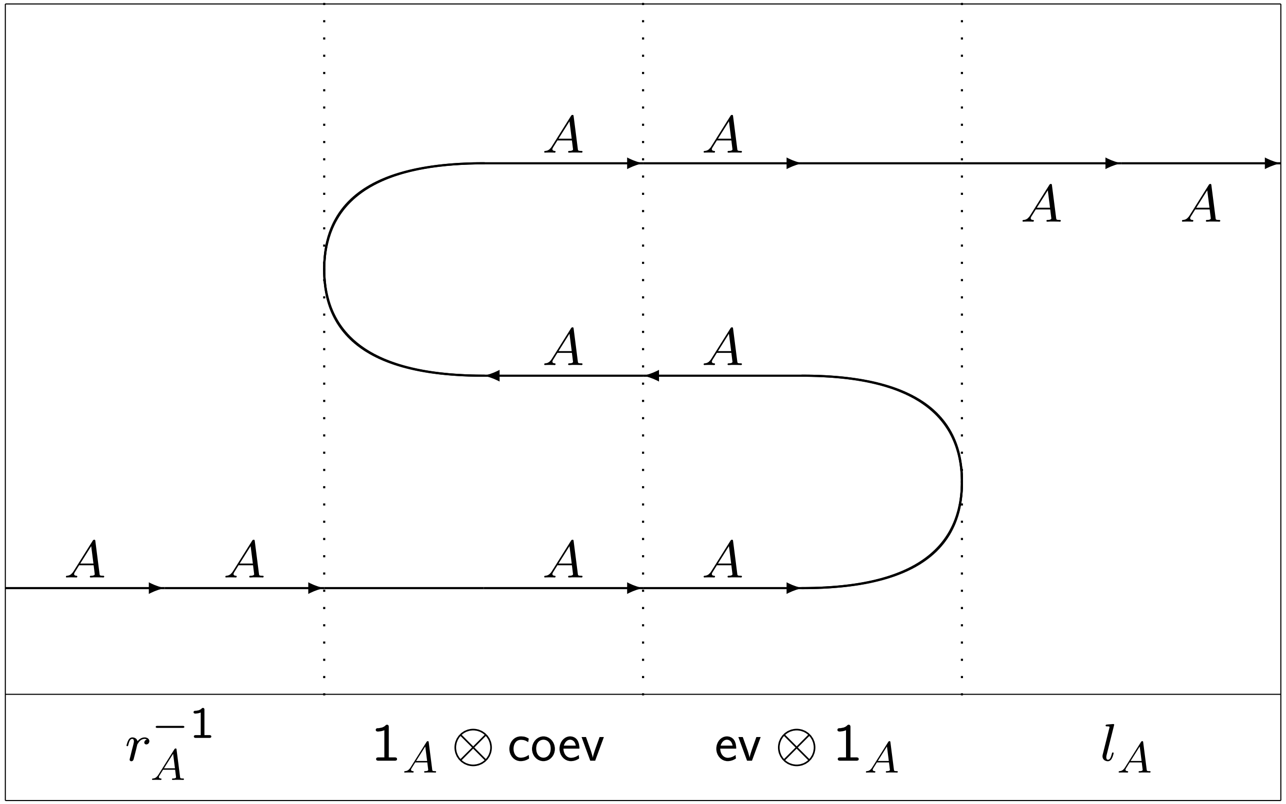

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First yanking condition

$$x' \overset{r_A^{-1}}{\longmapsto} x' \otimes 1 \overset{1_A \otimes \mathsf{coev}}{\longmapsto} x' \otimes \sum_{x \in X} \delta_x \otimes x \overset{\mathsf{ev} \otimes 1_A}{\longmapsto} \sum_{x \in X} x'\delta_x \otimes x = 1 \otimes x' \overset{l_A}{\longmapsto} x'$$

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A \otimes A & \xrightarrow{\mathsf{coev}_A \otimes \mu} & A^* \otimes A \otimes A^* & \xrightarrow{1_{A^*} \otimes \Delta \otimes 1_{A^*}} & A^* \otimes A \otimes A \otimes A^* \\
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Currying and braiding in compact closed categories

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Augmented quasigroup: Augmented magma $(A, \mu, \Delta, \varepsilon)$
for which $(A, \rho, \Delta, \varepsilon)$ and $(A, \lambda, \Delta, \varepsilon)$ are augmented magmas.

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Works equally well for a finite quasigroup $(G, \cdot, /, \backslash)$.

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The dual group of a finite abelian group G . . .

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Example: Group S_3 with character table

S_3	1	t	c
χ_1	1	1	1
χ_2	1	-1	1
θ	2	0	-1

A dual quasigroup of a finite group G . . .

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$\otimes$	χ_1	χ_2	θ
χ_1	χ_1	χ_2	θ
χ_2	χ_2	χ_1	θ
θ	θ	θ	$\chi_1 + \chi_2 + \theta$

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Example: Group S_3 with character table

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$\widetilde{S_3}$	χ_1	χ_2	θ_1	θ_2	θ_3	θ_4
χ_1	χ_1	χ_2	θ_1	θ_2	θ_3	θ_4
χ_2	χ_2	χ_1	θ_2	θ_1	θ_4	θ_3
θ_1	θ_1	θ_2	θ_3	θ_4	χ_1	χ_2
θ_2	θ_2	θ_1	θ_4	θ_3	χ_2	χ_1
θ_3	θ_3	θ_4	χ_1	χ_2	θ_1	θ_2
θ_4	θ_4	θ_3	χ_2	χ_1	θ_2	θ_1

$\rightarrow$

$\otimes$	χ_1	χ_2	θ
χ_1	χ_1	χ_2	θ
χ_2	χ_2	χ_1	θ
θ	θ	θ	$\chi_1 + \chi_2 + \theta$

A dual quasigroup of a finite group G ...

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Example: Group S_3 with character table

S_3	1	t	c
χ_1	1	1	1
χ_2	1	-1	1
θ	2	0	-1

$\widetilde{S_3}$	χ_1	χ_2	θ_1	θ_2	θ_3	θ_4
χ_1	χ_1	χ_2	θ_1	θ_2	θ_3	θ_4
χ_2	χ_2	χ_1	θ_2	θ_1	θ_4	θ_3
θ_1	θ_1	θ_2	θ_3	θ_4	χ_1	χ_2
θ_2	θ_2	θ_1	θ_4	θ_3	χ_2	χ_1
θ_3	θ_3	θ_4	χ_1	χ_2	θ_1	θ_2
θ_4	θ_4	θ_3	χ_2	χ_1	θ_2	θ_1

$\rightarrow$

$\otimes$	χ_1	χ_2	θ
χ_1	χ_1	χ_2	θ
χ_2	χ_2	χ_1	θ
θ	θ	θ	$\chi_1 + \chi_2 + \theta$

E.g., $\theta(1)^3\langle \theta \otimes \theta | \theta \rangle = 8 = |\{\theta_1, \theta_2\}^2 \cup \{\theta_3, \theta_4\}^2|$.

Thank you for your attention!