Ω-groupoids and Ω-quasigroups

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Hence, we investigate Ω-groupoids and similar Ω-structures denoted by $(\mathcal{A}, E)$.
Identities and polynomial formulas with equality are formulated as particular lattice formulas in which the equality sign is replaced by E. Then an identity (formula) holds in $(\mathcal{A}, E)$ if the corresponding lattice formula is satisfied in Ω.

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We prove the equivalence of Ω-quasigroups with *Ω-equasigroups* in which the basic structure possesses three binary operations and the classical equasigroup identities are supposed to hold as particular lattice formulas.
We also investigate Ω-groupoids with a unit. We prove that *a unit is unique if the language contains a nullary operation.* Otherwise, *an Ω-groupoid might contain several units, which are equal up to the Ω-valued equality E.*

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Finally, we deal with (unique) solutions of equations $a \cdot x = b$ and $y \cdot a = b$ in the framework of Ω-quasigroups and Ω-groups. For an Ω-quasigroup (Ω-group) $(\mathcal{A}, E)$ we obtain solutions with respect (up to) the Ω-equality E - a kind of approximate solutions.

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$$Q1: \; y = x \cdot (x \backslash y);$$
$$Q2: \; y = x \backslash (x \cdot y);$$
$$Q3: \; y = (y/x) \cdot x;$$
$$Q4: \; y = (y \cdot x)/x.$$

If $(Q, \cdot)$ *is a quasigroup, then* $(Q, \cdot, \backslash, /)$ *is an equasigroup, where the additional binary operations* $\backslash$ *and* $/$ *are defined by:*

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– as a groupoid $(G, \cdot)$, i.e., in the language with a single binary operation $\cdot$.

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The collection μ_Ω of cuts of an Ω-valued function $\mu : X \to \Omega$ is a **closure system**, a complete lattice under the set-inclusion, consisting of subsets of X closed under set-intersections, containing also X.

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For every binary operation $*$ and unary $'$ in F, for all $a_1, a_2, b_1, b_2 \in Q$, and for every constant (nullary operation) $c \in F$

$$E(a_1, b_1) \wedge E(a_2, b_2) \leqslant E(a_1 * a_2, b_1 * b_2);$$
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$\mathcal{Q}$ is the **underlying, basic algebra** of $\overline{\mathcal{Q}}$.

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Lemma

If $(\mathcal{Q}, E)$ is an Ω-algebra and $p \in \Omega$, then the cut μ_p is a subalgebra of $\mathcal{Q}$ and the cut E_p is a congruence relation on μ_p.

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Let $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ (briefly $u = v$) be an identity in the type of an Ω-algebra $(\mathcal{Q}, E)$. We assume, as usual, that variables appearing in terms u and v are from $x_1, \ldots, x_n$ Then, $(\mathcal{Q}, E)$ **satisfies identity** $u = v$ if the following condition is fulfilled:

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Theorem

Let $(\mathcal{Q}, E)$ be an Ω-algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{Q}$. Then, $(\mathcal{Q}, E)$ satisfies all the identities in $\mathcal{F}$ if and only if for every $p \in \Omega$ the quotient algebra μ_p/E_p satisfies the same identities.

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Each of the formulas $a \cdot x = b$ and $y \cdot a = b$, $a, b \in Q$, x, y – variables, is a **linear equation over** $(\mathcal{Q}, E)$.

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Analogously, an equation $y \cdot a = b$ is **solvable over** $(\mathcal{Q}, E)$ if there is $d \in Q$ such that

$$\mu(a) \wedge \mu(b) \leqslant \mu(d) \wedge E(d \cdot a, b).$$

Elements c and d are **solutions of equations** $a \cdot x = b$ **and** $y \cdot a = b$, respectively **in** $(\mathcal{Q}, E)$.

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Theorem

Let $(\mathcal{Q}, E)$ be an Ω-groupoid. If equations $a \cdot x = b$ and $y \cdot a = b$, are E-uniquely solvable over $(\mathcal{Q}, E)$ for all $a, b \in Q$, then for every $p \in \Omega$ the quotient groupoid μ_p/E_p is a quasigroup.

We say that an Ω-groupoid $(\mathcal{Q}, E)$ is an **Ω-quasigroup**, if every equation of the form $a \cdot x = b$ or $y \cdot a = b$ is E-uniquely solvable over $(\mathcal{Q}, E)$.

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Theorem

Let $(\mathcal{Q}, E)$ be an Ω-groupoid. If for all $a, b \in Q$ and for every $p \leqslant \mu(a) \wedge \mu(b)$ the quotient groupoid μ_p / E_p is a quasigroup, then $(\mathcal{Q}, E)$ is an Ω-quasigroup.

Ω-equasigroup

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Let $\mathcal{Q} = (Q, \cdot, \backslash, /)$ be an algebra in the language with three binary operations, Ω a complete lattice and $E : Q^2 \to \Omega$ an Ω-valued compatible equality over $\mathcal{Q}$.

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Then, $(\mathcal{Q}, E)$ is an **Ω-equasigroup**, if identities $Q1, \ldots, Q4$ hold:

$Q1: \; y = x \cdot (x \backslash y);$

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Theorem

If $((Q, \cdot, \backslash, /), E)$ is an Ω-equasigroup, then for every $p \in \Omega$, the quotient structure μ_p / E_p is a classical equasigroup.

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The converse follows by the Axiom of Choice (AC).

Theorem

Let $((Q,\cdot),E)$ be an Ω-groupoid which is an Ω-quasigroup. Then the structure $((Q,\,\cdot\,,\backslash\,,/\,),E)$ is an Ω-equasigroup.

Example

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$\cdot$	a	b	c	d	e
a	b	c	a	a	e
b	a	b	c	d	e
c	c	a	b	b	e
d	d	a	b	b	e
e	e	e	e	e	a

Table 1

Example

$\cdot$	a	b	c	d	e
a	b	c	a	a	e
b	a	b	c	d	e
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d	d	a	b	b	e
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Table 1

Let $(Q, \cdot)$ be a groupoid given in Table 1.

Example

$\cdot$	a	b	c	d	e
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c	c	a	b	b	e
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e	e	e	e	e	a

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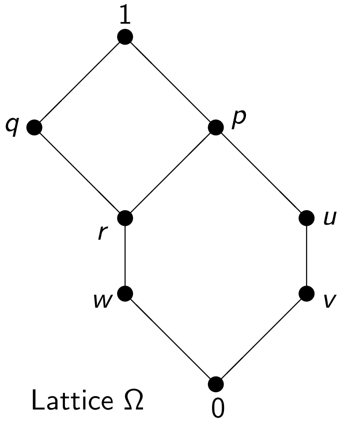
Let $(Q, \cdot)$ be a groupoid given in Table 1.

This is not a quasigroup, e.g., equation $a \cdot x = d$ does not have a solution in Q.

The lattice Ω is given by the diagram in Figure 1:



Lattice Ω

Figure 1

An Ω-valued equality is presented by Table 2.

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E	a	b	c	d	e
a	1	p	p	r	v
b	p	1	p	r	v
c	p	p	1	q	v
d	r	r	q	q	0
e	v	v	v	0	u

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E	a	b	c	d	e
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Table 2

The function $\mu : Q \to \Omega$ ($\mu(x) = E(x,x)$ for all $x \in Q$):

$$\mu = \begin{pmatrix} a & b & c & d & e \\ 1 & 1 & 1 & q & u \end{pmatrix}.$$

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Indeed, due to $\mu(a) \wedge \mu(d) = q$, this solution is element b, since the class $X = \{b\}$ is the unique solution of the equation $[a]_{E_q} \cdot X = [d]_{E_q}$ over the quasigroup μ_q / E_q (observe that $[d]_{E_q} = \{c, d\}$).

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Hence, $a \cdot b$ and d are E-equal with grade q.

Ω-loop and Ω-group

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An Ω-algebra $(\mathcal{G}, E)$ is an **Ω-group**, if the underlying algebra $\mathcal{G} = (G, \cdot, {}^{-1}, e)$ has a binary operation $\cdot$, a unary operation ${}^{-1}$, a constant e, and the following formulas hold:

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$$LG1: \ \mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E(x \cdot (y \cdot z), (x \cdot y) \cdot z);$$
$$LG2: \ \mu(x) \leqslant E(x \cdot e, x), \ \ \mu(x) \leqslant E(e \cdot x, x);$$
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Theorem

An Ω-algebra $((G, \cdot, {}^{-1}, e), E)$ is an Ω-group if and only if for every $p \in \Omega$, the quotient cut-subalgebra μ_p / E_p is a group.

An ***O*-loop** is an Ω-algebra $(\mathcal{Q}, E)$, where $\mathcal{Q} = (Q, \cdot, e)$ is a structure with a binary operation $\cdot$ and a constant e, $((Q, \cdot), E)$ is an Ω-quasigroup, $E(e, e) = 1$ and the formula $LG2$ holds.

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The proof of the following theorem depends on the Axiom of Choice (AC).

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An **Ω-semigroup** is an Ω-algebra $((Q, \cdot), E)$ where $(Q, \cdot)$ is a groupoid and the formula $LG1$ holds.

The proof of the following theorem depends on the Axiom of Choice (AC).

Theorem

Let $((Q, \cdot, e), E)$ be an Ω-algebra. There is a unary operation $^{-1}$ on Q such that $((Q, \cdot, {}^{-1}, e), E)$ is an Ω-group if and only if $((Q, \cdot), E)$ is an Ω-semigroup and $((Q, \cdot, e), E)$ an Ω-loop.

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If e is a neutral element in an Ω-groupoid $((Q,\cdot),E)$, then for every $x \in G$, $\mu(x) \leqslant \mu(e)$.
In addition, e is unique and this is also the neutral element in the underlying groupoid $(Q,\cdot)$.

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Let e be a weak neutral element in an Ω-groupoid $\overline{\mathcal{Q}} = (\mathcal{Q}, E)$, where $\mathcal{Q} = (Q, *)$ is a classical groupoid. Then for every $p \in \Omega$ such that $e \in \mu_p$, the class $[e]_{E_p}$ is a neutral element in the groupoid μ_p/E_p.

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A weak neutral element e in an Ω-groupoid $\overline{\mathcal{Q}} = (\mathcal{Q}, E)$ is an idempotent element in the underlying groupoid $\mathcal{Q}$.

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Proposition

If e_1 and e_2 are weak neutral elements in an Ω-groupoid $\overline{\mathcal{Q}}$, then

$$E(e_1, e_2) = \mu(e_1) \wedge \mu(e_2).$$

In addition, if $e_1 \neq e_2$, then $\mu(e_1)$ and $\mu(e_2)$ are not comparable elements of Ω.

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(jj) If e is a weak neutral element of $\overline{\mathcal{Q}}$, then for every $x \in Q$ there is $x^{-1_e} \in Q$ such that
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(*jjj*) $\mu^{-1}(\downarrow \mu(N)) = Q$.

Theorem

Let $\overline{\mathcal{Q}} = (\mathcal{Q}, E)$ *be a weak* Ω*-group. Then for every* $p \in \Omega$, μ_p / E_p *is a group in which the neutral class is* $[e]_{E_p}$ *for an* $e \in N$.

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Theorem

Let $\overline{\mathcal{Q}} = (\mathcal{Q}, E)$ be an Ω-groupoid with a nonempty set N of weak neutral elements. If every nonempty quotient μ_p/E_p, $p \in \Omega$, is a group whose neutral class is $[e]_{E_p}$ for an $e \in N$, then $\overline{\mathcal{Q}}$ is a weak Ω-group.

Example

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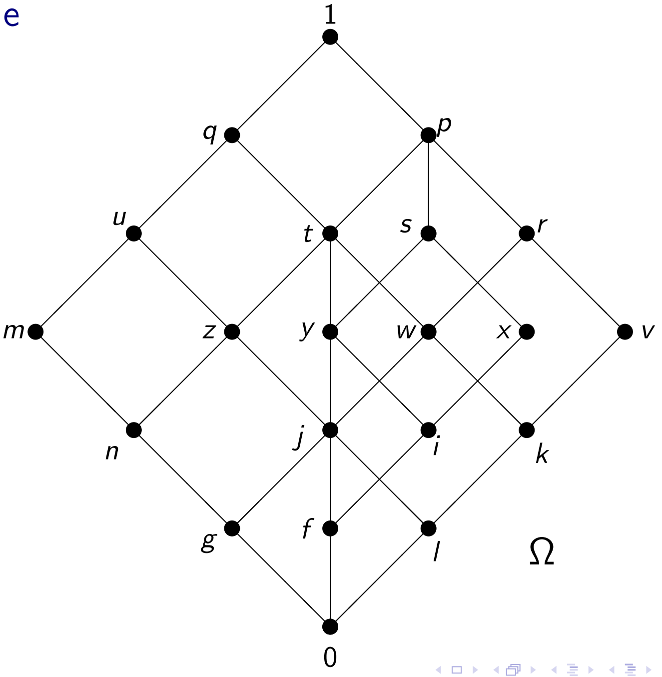

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e_2	e_2	e_2	a	b	c
a	a	a	e_1	c	b
b	b	b	c	e_1	a
c	c	c	b	a	e_2

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a	a	a	e_1	c	b
b	b	b	c	e_1	a
c	c	c	b	a	e_2

E	e_1	e_2	a	b	c
e_1	p	t	v	x	n
e_2	t	q	k	i	m
a	v	k	r	g	f
b	x	i	g	s	l
c	n	m	f	l	u

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c	c	c	b	a	e_2

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c	c	c	b	a	e_2

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$\mu_1/E_1 = \emptyset$, $\mu_p/E_p = \{\{e_1\}\}$, $\mu_q/E_q = \{\{e_2\}\}$,
$\mu_t/E_t = \{\{e_1, e_2\}\}$, $\mu_u/E_u = \{\{e_2\}, \{c\}\}$, $\mu_s/E_s = \{\{e_1\}, \{b\}\}$,
$\mu_r/E_r = \{\{e_1\}, \{a\}\}$, $\mu_m/E_m = \{\{e_2, c\}\}$,
$\mu_z/E_z = \{\{e_1, e_2\}, \{c\}\}$, $\mu_y/E_y = \{\{e_1, e_2\}, \{b\}\}$,
$\mu_x/E_x = \{\{e_1, b\}\}$, $\mu_w/E_w = \{\{e_1, e_2\}, \{a\}\}$, $\mu_v/E_v = \{\{e_1, a\}\}$,
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$\mu_g/E_g = \{\{e_1, e_2, c\}, \{a, b\}\}$, $\mu_f/E_f = \{\{e_1, e_2, b\}, \{a, c\}\}$,
$\mu_l/E_l = \{\{e_1, e_2, a\}, \{b, c\}\}$, $\mu_0/E_0 = \{\{e_1, e_2, a, b, c\}\}$.

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Clearly, e_1 and e_2 are weak neutral elements and the condition (jj) holds: $N = \{e_1,e_2\}$, $\mu(N) = \{p,q\}$ and $\mu^{-1}(\downarrow\{p,q\}) = Q$.

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Thank you!