Quasigroups for cryptography

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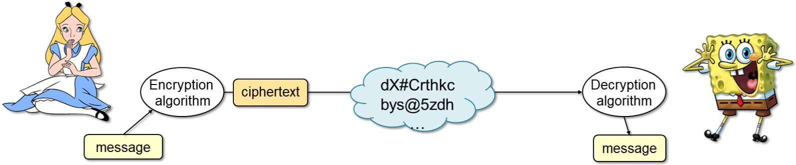
LOOPS 19

A typical everyday scenario



- Symmetric cryptography
  - Secrecy (encrypt the traffic)
  - Integrity of messages
- Public key cryptography
  - Exchange symmetric keys
  - Entity authentication
  - Non-repudiation
- Adversary
  - Can listen to the traffic (passive)
  - Can change the traffic (active)

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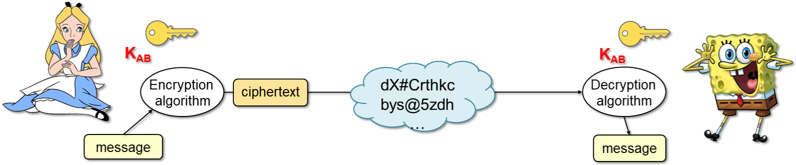

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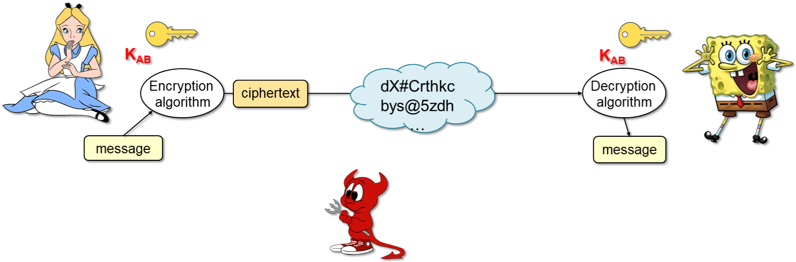

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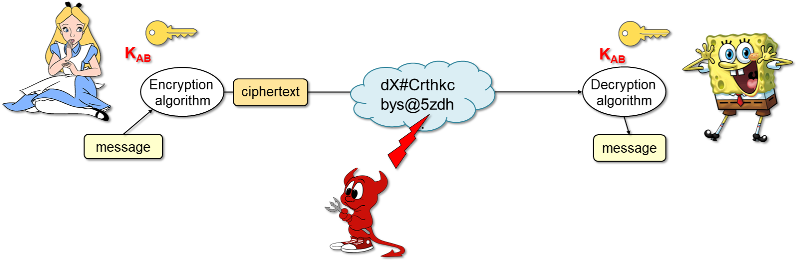

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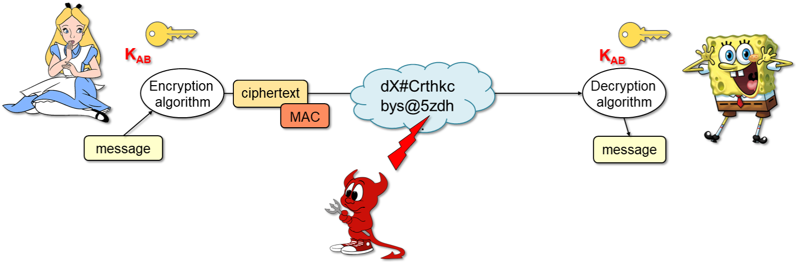

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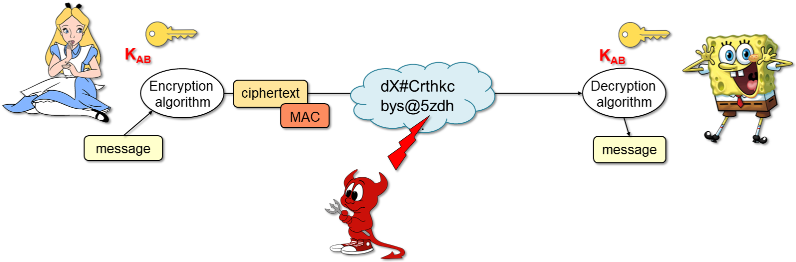

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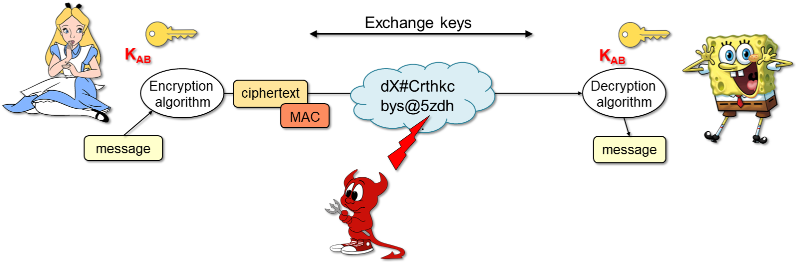

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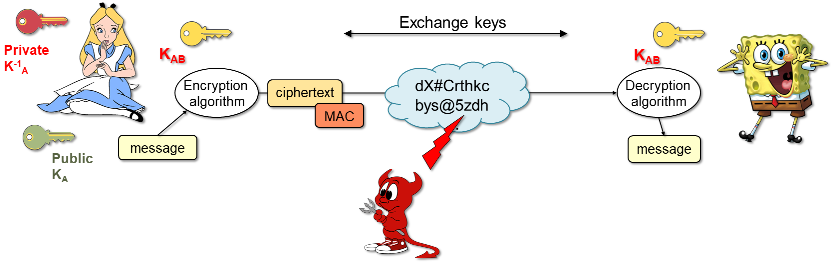

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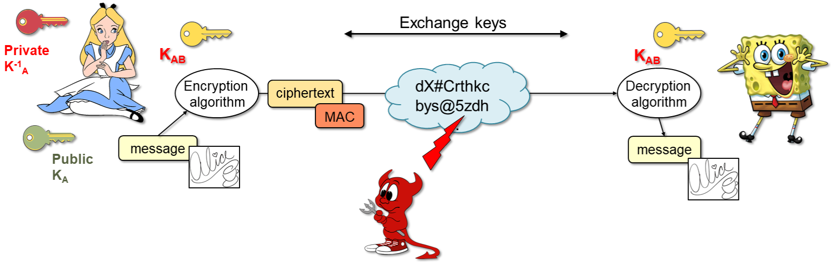

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Crypto comes in many flavors....(none of which is cryptocurrency)

- Symmetric cryptography
  - PRNGs
  - Stream ciphers
  - Block ciphers
  - hash functions
  - AEAD
  - Lightweight crypto
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Typical design choices in quasigroup based cryptography

- Quasigroups of order 4
  - Edon80, GAGE, InGAGE
- Quasigroups string transformations
  - All designs
- Mixing using Orthogonal latin squares
  - All symmetric designs
- Huge quasigroups from ARX operations
  - Edon-R, π-Cipher, BMW
- Multivariate Quadratic Quasigroups (MQQ)
  - MQQ-Sig, MQQ-Enc, IDS from quasigroups

$\bullet_0$	0	1	2	3
0	0	2	1	3
1	2	1	3	0
2	1	3	0	2
3	3	0	2	1

$\bullet_1$	0	1	2	3
0	1	3	0	2
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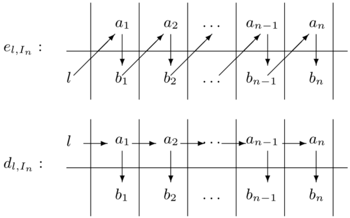

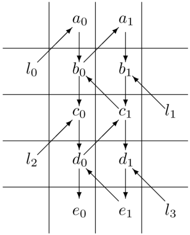

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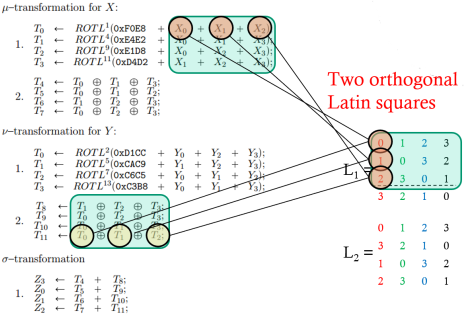

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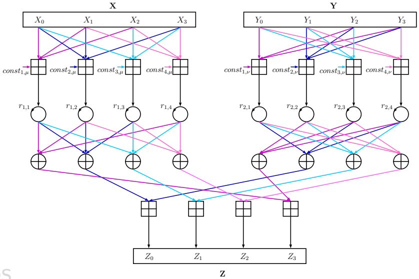

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MQQ of order 8

*	0	1	2	3	4	5	6	7
0	2	3	6	7	0	1	5	4
1	6	7	5	4	2	3	0	1
2	3	2	7	6	1	0	4	5
3	7	6	4	5	3	2	1	0
4	4	5	0	1	7	6	2	3
5	0	1	3	2	5	4	7	6
6	5	4	1	0	6	7	3	2
7	1	0	2	3	4	5	6	7

$$q = (q^{(1)}, q^{(2)}, q^{(3)}) : \mathbb{F}_2^6 \to \mathbb{F}_2^3$$

$$q^{(1)} = x_1 + x_3 + y_2 + x_1y_2 + x_1y_3,$$
$$q^{(2)} = 1 + x_3 + x_1y_2 + y_3 + x_1y_3,$$
$$q^{(3)} = x_2 + y_1 + x_1y_2 + x_3y_3 + y_2y_3$$

The status of these cryptosystems?

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Standard security evaluation in symmetric key crypto

Security from ideal primitives $+$ Cryptanalysis of real constructions

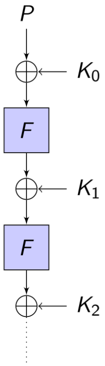

Figure: Typical Design

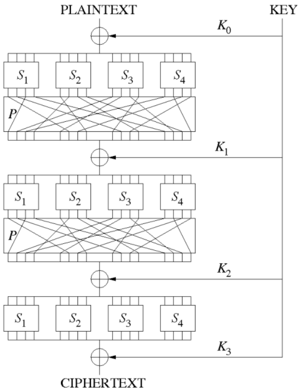

Figure: Classical SPN

Linear cryptanalysis

- Invented by Mitsuru Matsui 1993
- Main idea:
  1. approximate the non-linear parts of the cipher by a linear relation between plaintext, (partial) keys and ciphertext
  2. calculate the probability that the relation holds
  3. if high enough, use as a distinguisher or a key recovery attack

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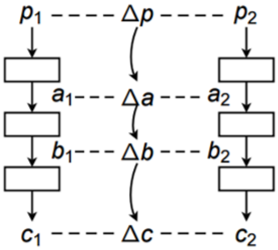
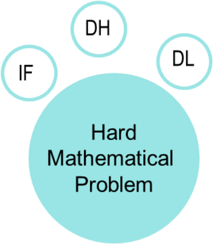
Differential cryptanalysis

- Invented by Biham and Shamir, late '80s
- Main idea:
  1. Observe the difference between two ciphertexts as a function of the difference between the plaintexts
  2. Find the highest probability differential through several rounds
  3. if high enough, use as a distinguisher or a key recovery attack



Standard security evaluation in public key crypto



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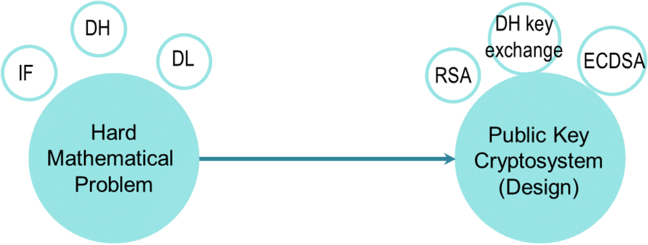

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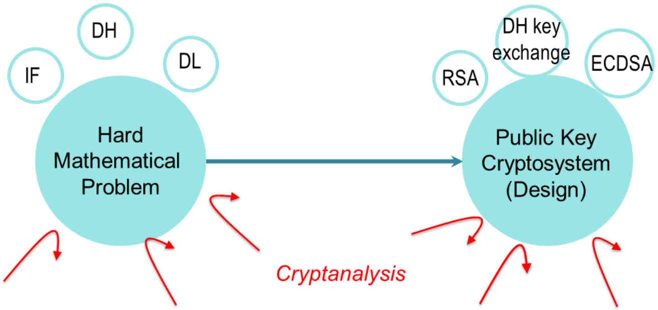

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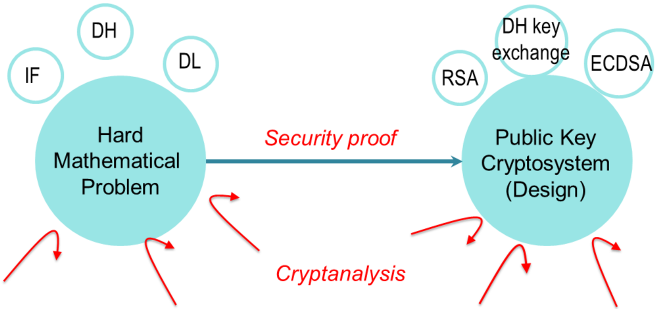

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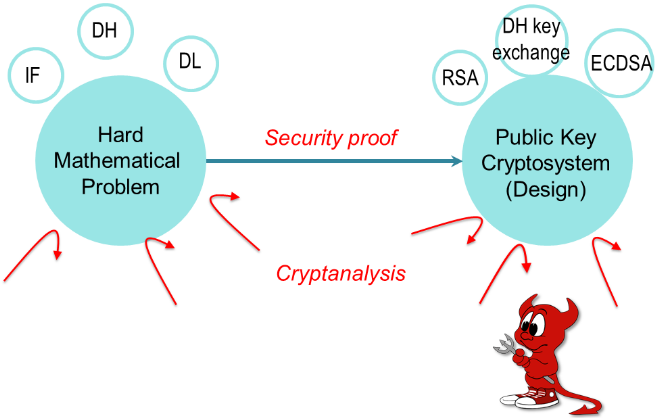

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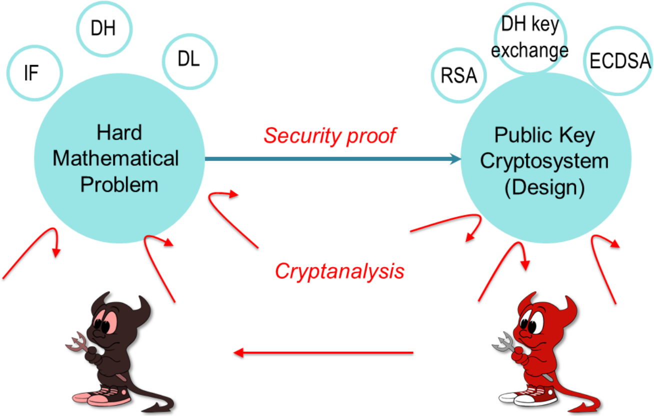

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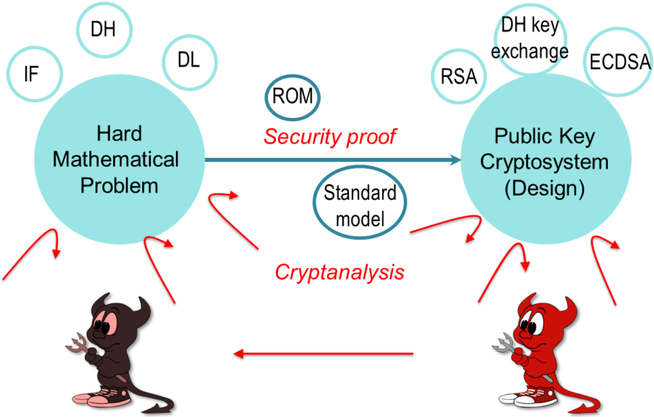

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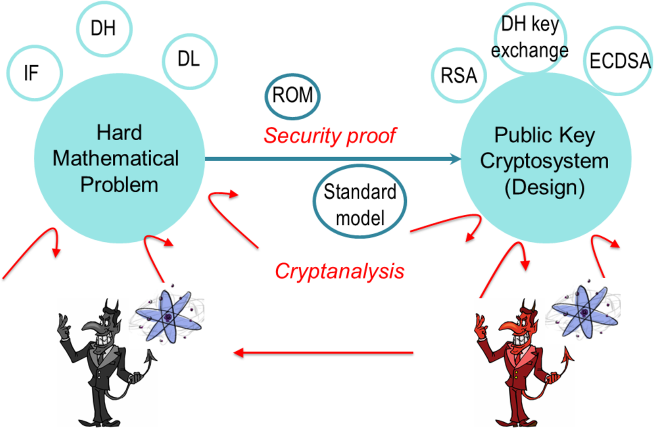

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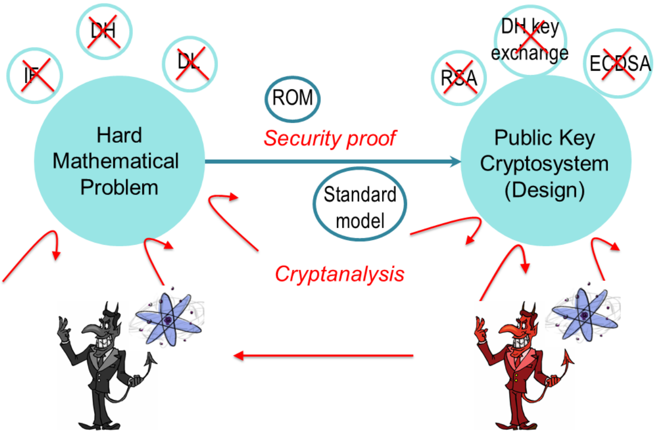

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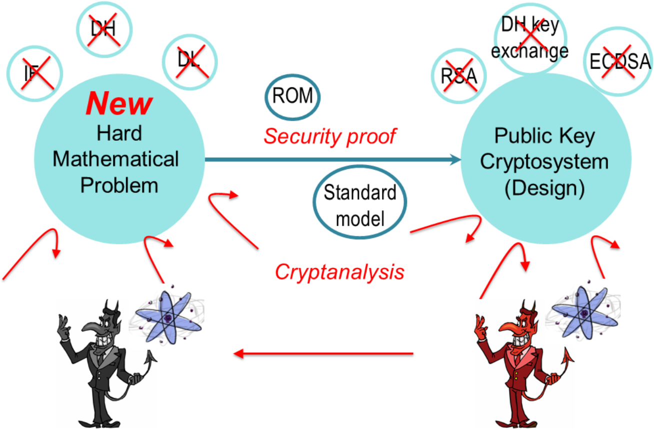

Linearity measures for (n, m)-functions

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Linearity of (n, m) functions $f : \mathbb{F}_q^n \to \mathbb{F}_q^m$:

$$\mathcal{L}(f) = \max_{w \in \mathbb{F}_q^m \setminus \{0\}, u \in \mathbb{F}_q^n} \left| \sum_{x \in \mathbb{F}_q^n} (-1)^{w^\intercal \cdot f(x) + u^\intercal \cdot x} \right|$$

Nonlinearity of (n, m) functions f:

$$\mathcal{N}(f) = (q-1)(q^{n-1} - \frac{1}{q}\mathcal{L}(f)).$$

$w \in \mathbb{F}_q^n$ - **linear structure** of f if

$$D_w f(x) = f(x+w) - f(x) = f(w) - f(0) \quad \forall\, x \in \mathbb{F}_q^n.$$

Linear space of f - generated by the linear structures of f.

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[Nyberg92] **Quadratic form f:**

- $x^{\intercal}\mathfrak{F}x$, $\mathrm{Rank}(\mathfrak{F}) = r$.
- $\mathrm{Ker}(\mathfrak{F})$ - **linear space of f.**

$$\mathcal{L}(f) = q^{n-\frac{r}{2}}$$

[Nyberg92] **Quadratic (n, m)-function f:**

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Maximum nonlinearity:

- **Bent functions** - $\mathrm{Rank}(\mathfrak{F}_v) = n$, even n, $m \leqslant n/2$,
- **Almost bent (AB) functions** - $\mathrm{Rank}(\mathfrak{F}_v) = n - 1$, odd $n = m$.

Example 1:

f :

$$\begin{aligned} f_1 &= x_1x_2 + x_3 \\ f_2 &= x_1x_3 + x_2 + x_3 \\ f_3 &= x_2x_3 + x_1 + x_2 + x_3 \\ f_4 &= x_1x_2 \end{aligned}$$

$\mathcal{L}(f) = 2^3$

$(1,0,0,1)^\intercal \quad \cdot f$ is linear

f' :

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It is important to measure the size of!

Example 2: Oil & Vinegar

f :

$$f_1(x_1, x_2, x_3, x_4) = x_1x_3 + x_2x_4 + x_1x_2 + x_3$$
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Example 2: Oil & Vinegar

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(s,t)–linearity of quadratic (n,m) function f

Boura and Canteaut FSE13:

f is said to be **(s,t)–linear** if there exist linear subspaces $V \subset \mathbb{F}_q^n$ with $\mathrm{Dim}(V) = s$, $W \subset \mathbb{F}_q^m$ with $\mathrm{Dim}(W) = t$, s.t.

$$\forall\, w \in W, \;\; w^{\intercal} \cdot f \text{ is linear on all cosets } V \oplus a \text{ of } V.$$

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- for $w \in W$, $\quad D_{a,b}(w^{\intercal} \cdot f) = 0, \quad \forall\ a, b \in V$.

Example:

$$f_1(x_1, x_2, x_3, x_4) = x_1x_3 + x_2x_4 + x_1x_2 + x_3$$
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f is $(2, 2)$–linear,
$V = \langle (0, 0, 1, 0), (0, 0, 0, 1) \rangle$, $W = \langle (1, 0), (0, 1) \rangle$

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Strong (s,t)–linearity of quadratic (n,m) function f

f is said to be **strongly (s,t)–linear** if there exist subspaces $V \subset \mathbb{F}_q^n$ with $\mathrm{Dim}(V) = s$, $W \subset \mathbb{F}_q^m$ with $\mathrm{Dim}(W) = t$, s.t.

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- for $w \in W$

$$D_a(w^\intercal \cdot f) = const., \quad \forall\, a \in V.$$

Example:

f :

$$\begin{aligned} f_1 &= x_1x_2 + x_3 \\ f_2 &= x_1x_3 + x_2 + x_3 \\ f_3 &= x_2x_3 + x_1 + x_2 + x_3 \\ f_4 &= x_1x_2 \end{aligned}$$

strongly $(3,1)$–linear
$V = \mathbb{F}_2^3$
$W = \langle(1,0,0,1)\rangle$

f' :

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strongly $(3,2)$–linear
$V = \mathbb{F}_2^3$
$W = \langle(1,1,0,0),(1,0,1,1)\rangle$

Simple but important properties

- **Strong** (s, t)**–linearity**
  $\Rightarrow$ **Strong** $(s-1, t)$**–linearity**
  $\Rightarrow$ **Strong** $(s, t-1)$**–linearity**
- (s, t)**–linearity**
  $\Rightarrow$ $(s-1, t)$**–linearity**
  $\Rightarrow$ $(s, t-1)$**–linearity**
- **Strong** (s, t)**–linearity** $\Rightarrow$ (s, t)**–linearity**

New development in linear attacks

- Propagation of affine subspaces
  - linked to both (s, t)–linearity and strong (s, t)–linearity
  - asks for classification of functions with respect to these two properties
- Invariant subspace attacks
  - powerful against lightweight ciphers
  - explained only recently [Leander et al. '15, Beierle et al. '17, Beyne '18, Liu et al. '19]

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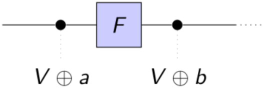

Consider a permutation formed by iterating a permutation F xored with a fixed round key K. Assume the round function maps a coset $V \oplus a$ to a coset $V \oplus b$

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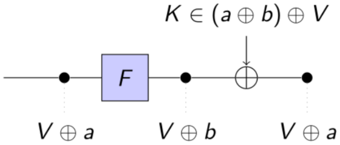

...and that the fixed round key K is in $V \oplus (a \oplus b)$.

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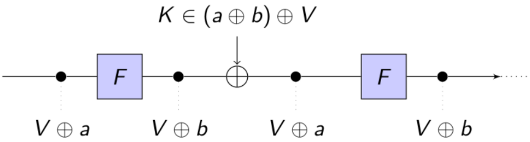
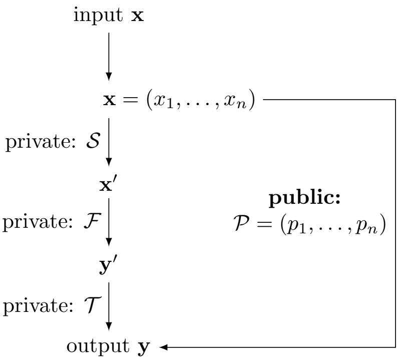


Then this process repeats itself.
Plaintexts in coset $V \oplus a$ are mapped to itself

Strong (s, t)–linearity vs $\mathcal{MQ}$ cryptography

Multivariate ($\mathcal{MQ}$) public key scheme: $\mathbb{F}_q^n \to \mathbb{F}_q^m$



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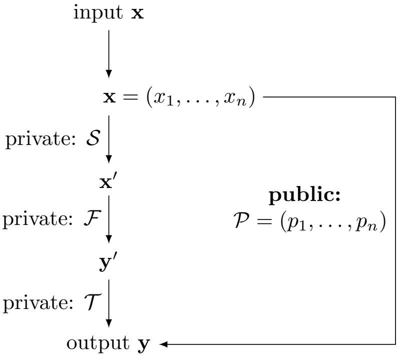

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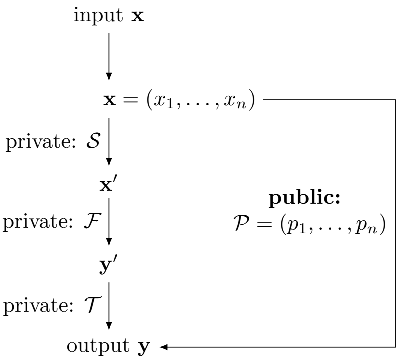

Public $\mathcal{P}$

$$p_1(x_1, \ldots, x_n)$$
$$p_2(x_1, \ldots, x_n)$$
$$\ldots$$
$$p_m(x_1, \ldots, x_n)$$

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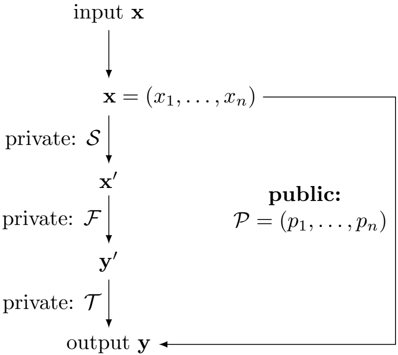

Public $\mathcal{P}$	Matrix form:
$p_1(x_1, \ldots, x_n)$	$x^\intercal \mathfrak{P}_1 x$
$p_2(x_1, \ldots, x_n)$	$x^\intercal \mathfrak{P}_2 x$
$\ldots$	$\ldots$
$p_m(x_1, \ldots, x_n)$	$x^\intercal \mathfrak{P}_m x$

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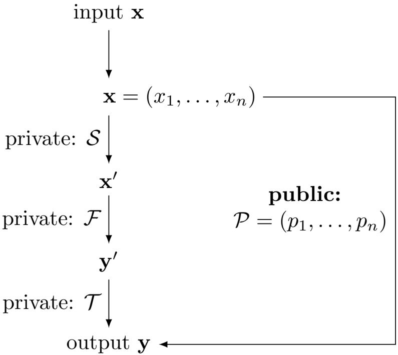

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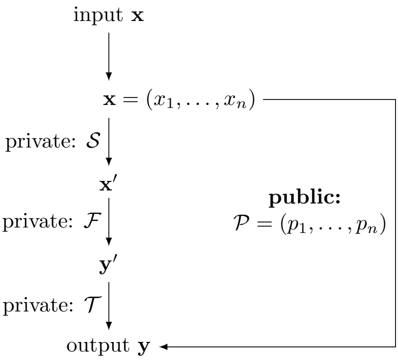
Symmetric matrices representing the quadratic part of the polynomials

Multivariate ($\mathcal{MQ}$) public key scheme: $\mathbb{F}_q^n \to \mathbb{F}_q^m$



Inverting $\mathcal{P}$ should be hard

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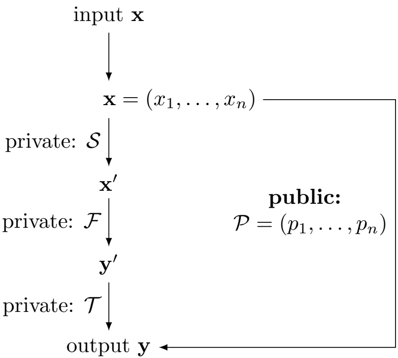

Inverting $\mathcal{P}$ should be hard

Underlying NP-complete problem

PoSSo:

Input:

$p_1, p_2, \ldots, p_m \in \mathbb{F}_q[x_1, \ldots, x_n]$

Question:

Find - if any - $(u_1, \ldots, u_n) \in \mathbb{F}_q^n$ st.

$$\begin{cases} p_1(u_1, \ldots, u_n) = & 0 \\ p_2(u_1, \ldots, u_n) = & 0 \\ \quad \ldots & \\ p_m(u_1, \ldots, u_n) = & 0 \end{cases}$$

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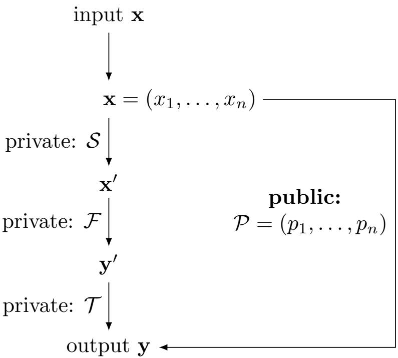

Attacks:

- MinRank
- Reconciliation/
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- Equivalent keys/
  Good keys
- Differential attacks

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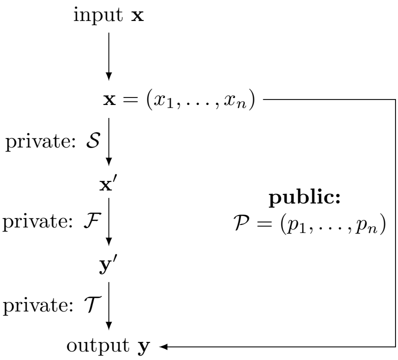

Attacks:

Linear subspaces!

- **MinRank**
- **Reconciliation/ Band separation attack**
- **Equivalent keys/ Good keys**
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MinRank $MR(n, m, r, M_1, \ldots, M_m)$

Input: $n, m, r \in \mathbb{N}$, and $M_1, \ldots, M_m \in \mathcal{M}_n(\mathbb{F}_q)$.
Question: Find – if any – a nonzero m-tuple $(\lambda_1, \ldots, \lambda_m) \in \mathbb{F}_q^m$ s.t.:

$$\text{Rank}\left(\sum_{i=1}^{m} \lambda_i M_i\right) \leqslant r.$$

[Courtois '01], [Buss & Shallit '99]

- **NP-hard!!!**, however,
- **Instances in $\mathcal{MQ}$ crypto** can be **much easier**, even **polynomial!**
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- Underlies the security of Code based cryptosystems in the rank metric
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Solving MinRank - Kipnis-Shamir modeling

$$\text{Rank}\left(\sum_{i=1}^{m} \lambda_i M_i\right) \leq r \;\Leftrightarrow\; \exists\, x^{(1)}, \ldots, x^{(n-r)} \in \text{Ker}\left(\sum_{i=1}^{m} \lambda_i M_i\right)$$

$$\begin{pmatrix} 1 & & & x_1^1 & \ldots & x_r^{(1)} \\ & \ddots & & \vdots & & \vdots \\ & & 1 & x_1^{(n-r)} & \ldots & x_r^{(n-r)} \end{pmatrix} \cdot \left(\sum_{i=1}^{m} \lambda_i M_i\right) = \mathbf{0}_{n \times n}.$$

$n\,(n-r)$ quadratic (bilinear) equations in $r\,(n-r) + m$ variables

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- Relinearization [Kipnis & Shamir '99]

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  - Complexity of F5 algorithm: $\mathcal{O}\left(\binom{n+d_{\mathrm{reg}}}{d_{\mathrm{reg}}}^{\omega}\right)$ [Faugère '02]

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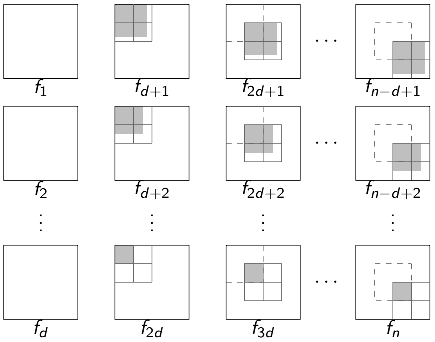
$$d_{reg} \leqslant \min(n_X, n_Y) + 1,$$

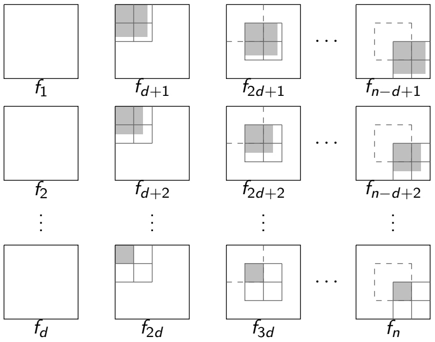
for bilinear system in X, Y blocks of variables of sizes n_X, n_Y.

Examples MinRank Cryptanalysis of MQQ [PKC '15]



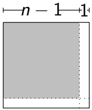
Hidden structure of secret $\mathcal{F}$

Examples MinRank Cryptanalysis of MQQ [PKC '15]



Hidden structure of secret $\mathcal{F}$

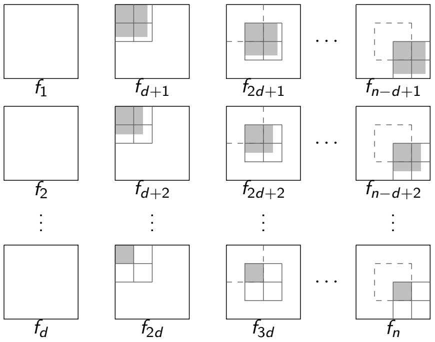
Recover structure



Find $(\lambda_1, \ldots, \lambda_m) \in (\mathbb{F}_q)^m$ s.t.

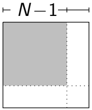
$$\text{rank}\left(\sum_{i=1}^{m} \lambda_i \mathfrak{P}_i\right) < n.$$

Examples MinRank Cryptanalysis of MQQ [PKC '15]



Hidden structure of secret $\mathcal{F}$

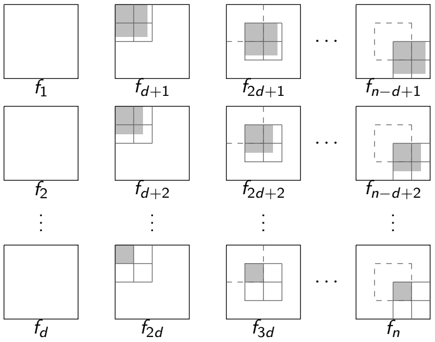
Recover structure



Find $(\lambda_1, \ldots, \lambda_{m'}) \in (\mathbb{F}_q)^{m'}$ s.t.

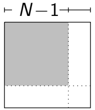
$$\operatorname{rank}\left(\sum_{i=1}^{m'} \lambda_i \mathfrak{P}'_i\right) < N.$$

Examples MinRank Cryptanalysis of MQQ [PKC '15]



Hidden structure of secret $\mathcal{F}$

Recover structure



Find $(\lambda_1, \ldots, \lambda_{m'}) \in (\mathbb{F}_q)^{m'}$ s.t.

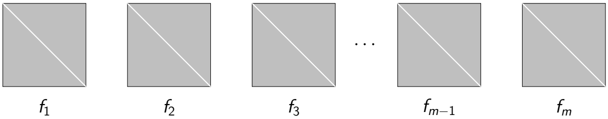
$$\text{rank}\left(\sum_{i=1}^{m'} \lambda_i \mathfrak{P}'_i\right) < N.$$

Complexity of MQQ Key recovery using MinRank with Gröbner bases:

$$\mathcal{O}(n^{10})$$

MinRank and Strong (s, t)–linearity

$f = (f_1, f_2, \ldots, f_m)$ - quadratic (n, m) function,
$\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_m$ - matrix representations of the coordinates of f.



MinRank and Strong (s, t)–linearity

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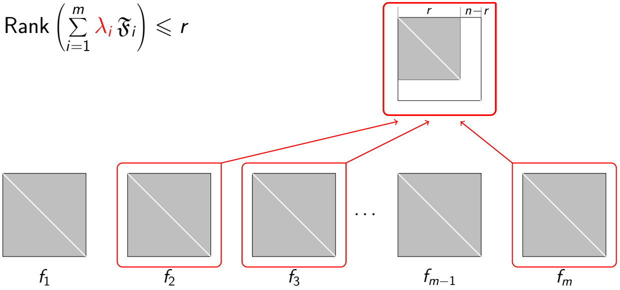
$\text{Rank}\left(\sum_{i=1}^{m} \lambda_i \mathfrak{F}_i\right) \leqslant r$



MinRank and Strong (s, t)–linearity

$f = (f_1, f_2, \ldots, f_m)$ - quadratic (n, m) function,
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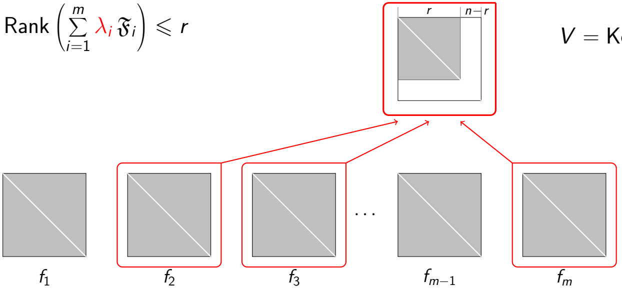
$\text{Rank}\left(\sum_{i=1}^{m} \lambda_i \mathfrak{F}_i\right) \leqslant r$

$V = \text{Ker}\left(\sum_{i=1}^{m} \lambda_i \mathfrak{F}_i\right)$



$W = \langle(\lambda_1, \ldots, \lambda_m)\rangle$

MinRank and Strong (s, t)–linearity

$f = (f_1, f_2, \ldots, f_m)$ - quadratic (n, m) function,
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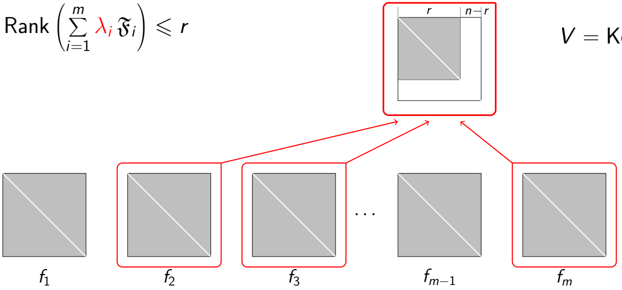


$W = \langle(\lambda_1, \ldots, \lambda_m)\rangle$

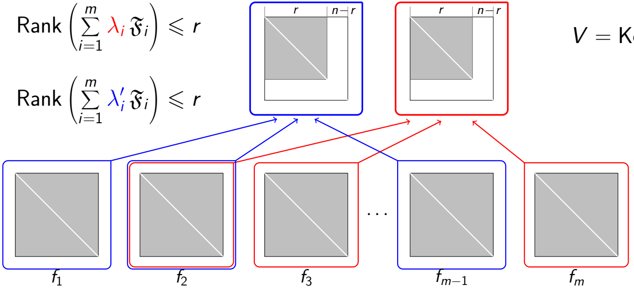
$MR(n, m, r, \mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_m)$ has a solution **iff** $\mathcal{L}(f) \geqslant q^{n-\frac{r}{2}}$
iff f **is strongly** $(n - r, 1)$**–linear**.

MinRank and Strong (s, t)–linearity

$f = (f_1, f_2, \ldots, f_m)$ - quadratic (n, m) function,
$\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_m$ - matrix representations of the coordinates of f.

$$\text{Rank}\left(\sum_{i=1}^{m} \lambda_i \mathfrak{F}_i\right) \leqslant r$$

$$\text{Rank}\left(\sum_{i=1}^{m} \lambda'_i \mathfrak{F}_i\right) \leqslant r$$



$$V = \text{Ker}\left(\sum_{i=1}^{m} \lambda_i \mathfrak{F}_i\right)$$

$$W = \langle(\lambda_1, \ldots, \lambda_m)\rangle$$

MinRank and Strong (s, t)–linearity

$f = (f_1, f_2, \ldots, f_m)$ - quadratic (n, m) function,
$\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_m$ - matrix representations of the coordinates of f.

$$\text{Rank}\left(\sum_{i=1}^{m} \lambda_i \mathfrak{F}_i\right) \leqslant r$$

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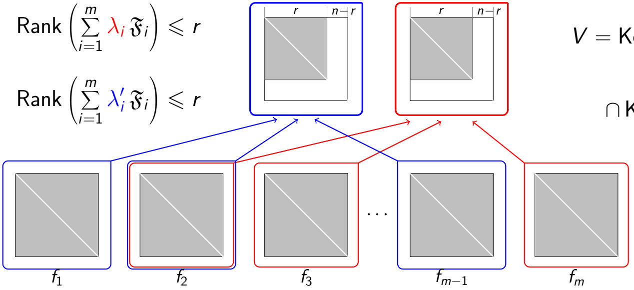

$$V = \text{Ker}\left(\sum_{i=1}^{m} \lambda_i \mathfrak{F}_i\right) \cap \text{Ker}\left(\sum_{i=1}^{m} \lambda'_i \mathfrak{F}_i\right)$$

$$W = \langle(\lambda_1, \ldots, \lambda_m), (\lambda_1, \ldots, \lambda_m)\rangle$$

Baby example

$$f_1(x_1,\ldots,x_6) = x_1x_3 + x_3x_5 + x_4x_5 + x_5 + x_4x_6 + x_6$$
$$f_2(x_1,\ldots,x_6) = x_1x_2 + x_1x_3 + x_1x_5 + x_1x_6 + x_2x_6 + x_3x_4 + x_3x_5 + x_3x_6 + x_4x_6 + x_6$$
$$f_3(x_1,\ldots,x_6) = x_1x_2 + x_2x_3 + x_1x_4 + x_3x_5 + x_4x_6 + x_5x_6$$

$$\mathfrak{F}_1 = \begin{bmatrix} 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{bmatrix} \qquad \mathfrak{F}_2 = \begin{bmatrix} 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \end{bmatrix} \qquad \mathfrak{F}_3 = \begin{bmatrix} 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \end{bmatrix}$$

Baby example

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$$\mathfrak{F}_1 = \begin{bmatrix} 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{bmatrix} \qquad \mathfrak{F}_2 = \begin{bmatrix} 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \end{bmatrix} \qquad \mathfrak{F}_3 = \begin{bmatrix} 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 \end{bmatrix}$$

$\mathrm{Rank}(\mathfrak{F}_3) = 4$

$\mathrm{Rank}(\mathfrak{F}_2 + \mathfrak{F}_3) = 4$

Baby example

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$$\mathfrak{F}_1 = \begin{bmatrix} 0&0&\mathbf{1}&0&0&0\\ 0&0&0&0&0&0\\ \mathbf{1}&0&0&0&\mathbf{1}&0\\ 0&0&0&0&\mathbf{1}&\mathbf{1}\\ 0&0&\mathbf{1}&\mathbf{1}&0&0\\ 0&0&0&\mathbf{1}&0&0 \end{bmatrix} \qquad \mathfrak{F}_2 = \begin{bmatrix} 0&\mathbf{1}&\mathbf{1}&0&\mathbf{1}&\mathbf{1}\\ \mathbf{1}&0&0&0&0&\mathbf{1}\\ \mathbf{1}&0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}\\ 0&0&\mathbf{1}&0&0&\mathbf{1}\\ \mathbf{1}&0&\mathbf{1}&0&0&0\\ \mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&0&0 \end{bmatrix} \qquad \mathfrak{F}_3 = \begin{bmatrix} 0&\mathbf{1}&0&\mathbf{1}&0&0\\ \mathbf{1}&0&\mathbf{1}&0&0&0\\ 0&\mathbf{1}&0&0&\mathbf{1}&0\\ \mathbf{1}&0&0&0&0&\mathbf{1}\\ 0&0&\mathbf{1}&0&0&\mathbf{1}\\ 0&0&0&\mathbf{1}&\mathbf{1}&0 \end{bmatrix}$$

$$\mathrm{Rank}(\mathfrak{F}_3) = 4$$
$$\mathrm{Rank}(\mathfrak{F}_2 + \mathfrak{F}_3) = 4$$

$$\mathrm{Ker}(\mathfrak{F}_3) \cap \mathrm{Ker}(\mathfrak{F}_2 + \mathfrak{F}_3) = \langle (0,1,0,1,1,0), (1,1,1,1,1,1) \rangle$$

Baby example $\quad f$ is $(2,2)$-strongly linear

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$$f_3(x_1,\ldots,x_6) = x_1x_2 + x_2x_3 + x_1x_4 + x_3x_5 + x_4x_6 + x_5x_6$$

$$\mathfrak{F}_1 = \begin{bmatrix} 0&0&\mathbf{1}&0&0&0\\ 0&0&0&0&0&0\\ \mathbf{1}&0&0&0&\mathbf{1}&0\\ 0&0&0&0&\mathbf{1}&\mathbf{1}\\ 0&0&\mathbf{1}&\mathbf{1}&0&0\\ 0&0&0&\mathbf{1}&0&0 \end{bmatrix} \qquad \mathfrak{F}_2 = \begin{bmatrix} 0&\mathbf{1}&\mathbf{1}&0&\mathbf{1}&\mathbf{1}\\ \mathbf{1}&0&0&0&0&\mathbf{1}\\ \mathbf{1}&0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}\\ 0&0&\mathbf{1}&0&0&\mathbf{1}\\ \mathbf{1}&0&\mathbf{1}&0&0&0\\ \mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&0&0 \end{bmatrix} \qquad \mathfrak{F}_3 = \begin{bmatrix} 0&\mathbf{1}&0&\mathbf{1}&0&0\\ \mathbf{1}&0&\mathbf{1}&0&0&0\\ 0&\mathbf{1}&0&0&\mathbf{1}&0\\ \mathbf{1}&0&0&0&0&\mathbf{1}\\ 0&0&\mathbf{1}&0&0&\mathbf{1}\\ 0&0&0&\mathbf{1}&\mathbf{1}&0 \end{bmatrix}$$

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$\dim(W) = 2$ $\qquad$ $\dim(V) = 2$

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$$\mathfrak{F}_1 = \begin{bmatrix} 0&0&\mathbf{1}&0&0&0\\ 0&0&0&0&0&0\\ \mathbf{1}&0&0&0&\mathbf{1}&0\\ 0&0&0&0&\mathbf{1}&\mathbf{1}\\ 0&0&\mathbf{1}&\mathbf{1}&0&0\\ 0&0&0&\mathbf{1}&0&0 \end{bmatrix} \quad \mathfrak{F}_2 = \begin{bmatrix} 0&\mathbf{1}&\mathbf{1}&0&\mathbf{1}&\mathbf{1}\\ \mathbf{1}&0&0&0&0&\mathbf{1}\\ \mathbf{1}&0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}\\ 0&0&\mathbf{1}&0&0&\mathbf{1}\\ \mathbf{1}&0&\mathbf{1}&0&0&0\\ \mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&0&0 \end{bmatrix} \quad \mathfrak{F}_3 = \begin{bmatrix} 0&\mathbf{1}&0&\mathbf{1}&0&0\\ \mathbf{1}&0&\mathbf{1}&0&0&0\\ 0&\mathbf{1}&0&0&\mathbf{1}&0\\ \mathbf{1}&0&0&0&0&\mathbf{1}\\ 0&0&\mathbf{1}&0&0&\mathbf{1}\\ 0&0&0&\mathbf{1}&\mathbf{1}&0 \end{bmatrix}$$

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$\dim(W) = 2$ $\qquad \dim(V) = 2$

Two MinRank problems with common kernel

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$$f_1(x_1,\ldots,x_6) = x_1x_3 + x_3x_5 + x_4x_5 + x_5 + x_4x_6 + x_6$$
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$\dim(W) = 2$ $\qquad$ $\dim(V) = 2$

After change of variables :

$$f_1(x_1,\ldots,x_6) = x_1x_4 + x_1x_6 + x_2x_3 + x_3x_4 + x_3x_5 + x_4x_5$$
$$f_2(x_1,\ldots,x_6) = x_1x_2 + x_1x_4 + x_2x_3$$
$$f_3(x_1,\ldots,x_6) = x_1x_3 + x_1x_4 + x_2x_3 + x_3x_4$$

MinRank and Strong (s, t)–linearity

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f is strongly (s, t)–linear

iff

$MR(n, m, n - s, \mathfrak{F}_1, \ldots, \mathfrak{F}_m)$
has t independent solutions $w_1, w_2, \ldots, w_t \in \mathbb{F}_q^m$ s.t.

$$\mathrm{Dim}\Big(\bigcap_i \mathrm{Ker}(\mathfrak{F}_{w_i})\Big) \geqslant s$$

MinRank and Strong (s,t)–linearity

$f = (f_1, f_2, \ldots, f_m)$ - quadratic (n, m) function,
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Simultaneous MinRank

Strong (s, t)–linearity v.s. $\mathcal{MQ}$ crypto

Simultaneous MinRank - Reveal strong (s, t) - linearity

Strong (s, t)–linearity v.s. $\mathcal{MQ}$ crypto

Simultaneous MinRank - Reveal strong (s, t) - linearity

Strongly (s, t)-linear
w.r.t. **unknown V, W**

Public $\mathcal{P}$

Strong (s, t)–linearity v.s. $\mathcal{MQ}$ crypto

Simultaneous MinRank - Reveal strong (s, t) - linearity

Strongly (s, t)-linear
w.r.t. **unknown V, W**

Goal: $\mathcal{P} \longrightarrow \mathcal{P}'$

Strongly (s, t)-linear
w.r.t. **standard basis**

Strong (s,t)–linearity v.s. $\mathcal{MQ}$ crypto

Simultaneous MinRank - Reveal strong (s,t) - linearity

Strongly (s,t)-linear
w.r.t. **unknown V, W**

Goal: $$\mathcal{P} = T' \circ \mathcal{P}' \circ S'$$

Strongly (s,t)-linear
w.r.t. **standard basis**

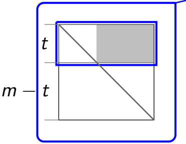
Strong (s, t)–linearity v.s. $\mathcal{MQ}$ crypto

Simultaneous MinRank - Reveal strong (s, t) - linearity

Strongly (s, t)-linear
w.r.t. **unknown V, W**

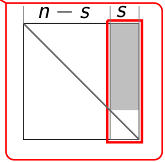
$$\mathcal{P} \; = \; T' \circ \mathcal{P}' \circ S'$$

Transform output space



Strongly (s, t)-linear
w.r.t. **standard basis**

Transform input space



Strong (s,t)–linearity v.s. $\mathcal{MQ}$ crypto

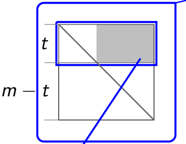
Simultaneous MinRank - Reveal strong (s,t) - linearity

Strongly (s,t)-linear
w.r.t. **unknown** V, W

$$\mathcal{P} \;=\; T' \circ \mathcal{P}' \circ S'$$

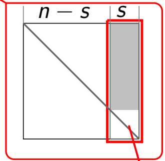
Transform output space

Transform input space



Strongly (s,t)-linear
w.r.t. **standard basis**

[Separate the spaces V, W]



basis of W

basis of V

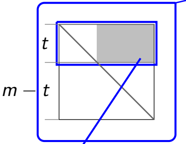
Strong (s, t)–linearity v.s. $\mathcal{MQ}$ crypto

Simultaneous MinRank - Reveal strong (s, t) - linearity

Strongly (s, t)-linear
w.r.t. **unknown** V**,** W

$$\mathcal{P} \;=\; T' \circ \mathcal{P}' \circ S'$$

Transform output space

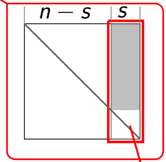


basis of W

Strongly (s, t)-linear
w.r.t. **standard basis**

[Separate the spaces V, W]

Transform input space



basis of V

(S', T') - **strong** (s, t) - **separation key**

Generic separation key attack for $\mathcal{MQ}$ cryptosystems

Repeat until

(1) Determine the existence of a strong (s, t) separation key

(2) Recover the linear spaces determined by the key

All structure of Central map $\mathcal{F}'$ revealed

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Repeat until

(1) Determine the existence of a strong (s, t) separation key

(2) **Recover the linear space determined by the key**

All structure of Central map $\mathcal{F}'$ revealed

(2) $\Leftrightarrow$ solving **Simultaneous MinRank**:

$$\mathfrak{P}_{w^{(i)}} \cdot v^{(j)} = 0,\ i \in \{1, \ldots, t\},\ j \in \{1, \ldots, s\},$$

in the unknown: - basis vectors $w^{(i)}$ of the space W,
- basis vectors $v^{(j)}$ of the space V.

Generic separation key attack for $\mathcal{MQ}$ cryptosystems

Repeat until

(1) Determine the existence of a strong (s, t) separation key

(2) **Recover the linear space determined by the key**

All structure of Central map $\mathcal{F}'$ revealed

Complexity:

$$\mathcal{O}\left(\binom{(n-s)s + (m-t)t + d_{reg}}{d_{reg}}^{\omega}\right)$$

$d_{reg} = \min\{(n-s)s, (m-t)t\} + 1$,
$2 \leqslant \omega \leqslant 3$ - linear algebra constant.

Generic separation key attack for $\mathcal{MQ}$ cryptosystems

Repeat until

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$2 \leqslant \omega \leqslant 3$ - linear algebra constant.

Not polynomial - time!

Repeat until

(1) Determine the existence of a strong (s, t) separation key

(2) **Recover the linear space determined by the key**

All structure of Central map $\mathcal{F}'$ revealed

(2) $\Leftrightarrow$ solving:

$$\mathfrak{P}_{w^{(i)}} \cdot v^{(j)} = 0, \; i \in \{1, \ldots, t\}, \; j \in \{1, \ldots, s\},$$

in the unknown: - basis vectors $w^{(i)}$ of the space W,
- basis vectors $v^{(j)}$ of the space V.

Generic separation key attack for $\mathcal{MQ}$ cryptosystems

Improved Min-Max strategy

Repeat until

(1) Determine the existence of a strong (s, t) separation key

(2) **Recover the linear space determined by the key**

All structure of Central map $\mathcal{F}'$ revealed

(2) $\Leftrightarrow$ solving: The quadratic:

$$\mathfrak{P}_{w^{(i)}} \cdot v^{(j)} = 0, \ i \in \{1, \ldots, c_1\}, \ j \in \{1, \ldots, c_2\}\},$$

in the unknown: - basis vectors $w^{(i)}$ of the space W,
- basis vectors $v^{(j)}$ of the space V.

Repeat until

(1) Determine the existence of a strong (s, t) separation key

(2) **Recover the linear space determined by the key**

All structure of Central map $\mathcal{F}'$ revealed

(2) $\Leftrightarrow$ solving: And then the linear:

$$\mathfrak{P}_{w^{(i)}} \cdot v^{(j)} = 0, \; i \in \{c_1+1, ..., t\}, \; j \in \{1, ..., c_2\},$$

$$\mathfrak{P}_{w^{(i)}} \cdot v^{(j)} = 0, \; i \in \{1, ..., c_1\}, \; j \in \{c_2+1, ..., s\},$$

in the unknown: - basis vectors $w^{(i)}$ of the space W,

- basis vectors $v^{(j)}$ of the space V.

Repeat until

(1) Determine the existence of a strong (s, t) separation key

(2) **Recover the linear space determined by the key**

All structure of Central map $\mathcal{F}'$ revealed

Complexity:

$$\mathcal{O}\left(\binom{(n-s)c_2 + (m-t)c_1 + d_{reg}}{d_{reg}}^{\omega}\right)$$

$d_{reg} = \min\{(n-s)c_2, (m-t)c_1\} + 1$,
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$2 \leqslant \omega \leqslant 3$ - linear algebra constant.

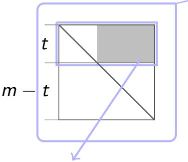
Polynomial - time!

An improved attack

Simultaneous MinRank - Reveal strong (s, t) - linearity

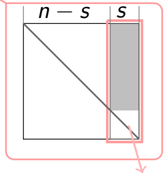
$$\mathcal{P} \;=\; T' \circ \mathcal{P}' \circ S'$$

Transform output space



basis of W

Transform input space



basis of V

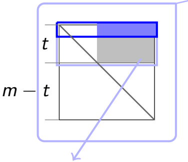
An improved attack

Simultaneous MinRank - Reveal strong (s, t) - linearity

$$\mathcal{P} = T' \circ \mathcal{P}' \circ S'$$

Transform output space

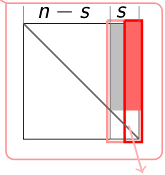
Transform input space



basis of W

Find only c_1 vectors from W

Find only c_2 vectors from V



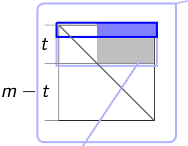
basis of V

An improved attack

Simultaneous MinRank - Reveal strong (s,t) - linearity

$$\mathcal{P} \;=\; T' \circ \mathcal{P}' \circ S'$$

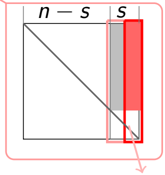
Transform output space



basis of W

Find only c_1 vectors from W
Find only c_2 vectors from V

Transform input space



basis of V

Choose c_1 and c_2 s.t.
the system has unique solution

First strong (c_2, c_1) - linearity
then strong (c_2, t) - linearity
then strong (s, t) - linearity

An improved attack

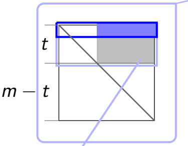
Simultaneous MinRank - Reveal strong (s,t) - linearity

Type of **good key** [Thomae-Wolf '12]
with "good enough" structure

$$\mathcal{P} \;=\; T' \circ \mathcal{P}' \circ S'$$

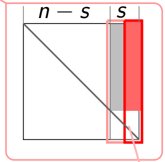
Transform output space

Transform input space



basis of W

← Find only c_1 vectors from W
Find only c_2 vectors from V →



basis of V

Choose c_1 and c_2 s.t.
the system has unique solution

First strong (c_2, c_1) - linearity
then strong (c_2, t) - linearity
then strong (s, t) - linearity

Recall: (s,t)–linearity of quadratic (n,m) function f

Boura and Canteaut FSE13:

f is said to be (s,t)**–linear** if there exist linear subspaces $V \subset \mathbb{F}_q^n$ with $\mathrm{Dim}(V) = s$, $W \subset \mathbb{F}_q^m$ with $\mathrm{Dim}(W) = t$, s.t.

$$\forall\, w \in W,\;\; w^{\intercal} \cdot f \text{ is linear on all cosets of } V.$$

- f_W corresponding to all $w^{\intercal} \cdot f$, $w \in W$ can be written as

$$f_W(x,y) = M(x) \cdot y + G(x)$$

where $\mathbb{F}_q^n = U \oplus V$, $G : U \to \mathbb{F}_q^t$ and $M(x)$ is a $t \times s$ matrix with rows - components of linear functions over U.

Baby example UOV

$$f_1(x_1,\dots,x_6) = x_1x_2 + x_2x_4 + x_3x_6 + x_4x_6 + x_5x_6 + x_6$$
$$f_2(x_1,\dots,x_6) = x_1x_4 + x_3x_4 + x_3x_6 + x_4x_6 + x_6$$
$$f_3(x_1,\dots,x_6) = x_2x_3 + x_3x_5 + x_2x_4 + x_2x_6 + x_4x_5 + x_1x_6 + x_4x_6 + x_5x_6$$

Baby example UOV

$$f_1(x_1, \ldots, x_6) = x_1x_2 + x_2x_4 + x_3x_6 + x_4x_6 + x_5x_6 + x_6$$
$$f_2(x_1, \ldots, x_6) = x_1x_4 + x_3x_4 + x_3x_6 + x_4x_6 + x_6$$
$$f_3(x_1, \ldots, x_6) = x_2x_3 + x_3x_5 + x_2x_4 + x_2x_6 + x_4x_5 + x_1x_6 + x_4x_6 + x_5x_6$$

$\overline{S}'$: $x_4 \to x_4 + x_6$
$x_2 \to x_2 + x_5$

Baby example UOV

$$f_1(x_1,\ldots,x_6) = x_1x_2 + x_2x_4 + x_3x_6 + x_4x_6 + x_5x_6 + x_6$$
$$f_2(x_1,\ldots,x_6) = x_1x_4 + x_3x_4 + x_3x_6 + x_4x_6 + x_6$$
$$f_3(x_1,\ldots,x_6) = x_2x_3 + x_3x_5 + x_2x_4 + x_2x_6 + x_4x_5 + x_1x_6 + x_4x_6 + x_5x_6$$

$$\overline{S}': \begin{array}{l} x_4 \to x_4 + x_6 \\ x_2 \to x_2 + x_5 \end{array}$$

After change of variables :

$$f_1(x_1,\ldots,x_6) = x_1x_2 + x_1x_5 + x_2x_4 + x_2x_6 + x_4x_5 + x_3x_6 + x_4x_6$$
$$f_2(x_1,\ldots,x_6) = x_1x_4 + x_1x_6 + x_3x_4 + x_4x_6$$
$$f_3(x_1,\ldots,x_6) = x_2x_3 + x_2x_4 + x_4x_6 + x_1x_6 + x_6$$

Baby example UOV

$$f_1(x_1,\ldots,x_6) = x_1x_2 + x_2x_4 + x_3x_6 + x_4x_6 + x_5x_6 + x_6$$
$$f_2(x_1,\ldots,x_6) = x_1x_4 + x_3x_4 + x_3x_6 + x_4x_6 + x_6$$
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$\overline{S}'$: $x_4 \to x_4 + x_6$ $\Leftrightarrow$ Linear on every coset of
$x_2 \to x_2 + x_5$ $\quad Span(\{(0,0,0,1,0,1),(0,1,0,0,1,0)\})$

After change of variables :

$$f_1(x_1,\ldots,x_6) = x_1x_2 + x_1x_5 + x_2x_4 + x_2x_6 + x_4x_5 + x_3x_6 + x_4x_6$$
$$f_2(x_1,\ldots,x_6) = x_1x_4 + x_1x_6 + x_3x_4 + x_4x_6$$
$$f_3(x_1,\ldots,x_6) = x_2x_3 + x_2x_4 + x_4x_6 + x_1x_6 + x_6$$

(s,t)–linearity v.s. $\mathcal{MQ}$ crypto

(s,t)-linear
w.r.t. **unknown** V**,** W

Public $\mathcal{P}$

(s, t)–linearity v.s. $\mathcal{MQ}$ crypto

(s, t)-linear
w.r.t. **unknown** V**,** W

Goal: $\mathcal{P} = T' \circ \mathcal{P}' \circ S'$

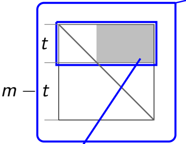
(s, t)-linear
w.r.t. **standard basis**

(s, t)–linearity v.s. $\mathcal{MQ}$ crypto

(s, t)-linear
w.r.t. **unknown** V**,** W

$$\mathcal{P} = T' \circ \mathcal{P}' \circ S'$$

Transform output space

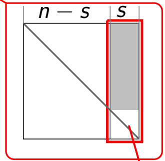


basis of W

(s, t)-linear
w.r.t. **standard basis**

[Separate the spaces V, W]

Transform input space



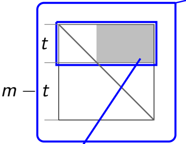
basis of V

(s,t)–linearity v.s. $\mathcal{MQ}$ crypto

(s,t)-linear
w.r.t. **unknown** V**,** W

$$\mathcal{P} = T' \circ \mathcal{P}' \circ S'$$

Transform output space

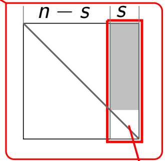


basis of W

(s,t)-linear
w.r.t. **standard basis**

[Separate the spaces V, W]

Transform input space



basis of V

(S', T') - (s,t) - **separation key**

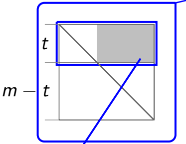
(s,t)–linearity v.s. $\mathcal{MQ}$ crypto

A realistic scenario - $(n, \mathbf{m})$ function is $(s, \mathbf{m})$ - linear

(s,t)-linear
w.r.t. **unknown** V, W

$$\mathcal{P} = T' \circ \mathcal{P}' \circ S'$$

Transform output space

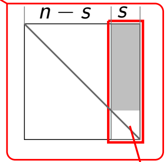


basis of W

(s,t)-linear
w.r.t. **standard basis**

[Separate the spaces V, W]

Transform input space



basis of V

(S', T') - (s,t) - separation key

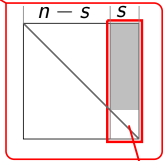
(s,t)–linearity v.s. $\mathcal{MQ}$ crypto

(s, m)-linear
w.r.t. **unknown** V

$$\mathcal{P} = \quad \mathcal{P}' \circ S'$$

(s, m)-linear
w.r.t. **standard basis**

Transform input space



basis of V

(s, t)–linearity v.s. $\mathcal{MQ}$ crypto

(s, m)-linear
w.r.t. **unknown** V

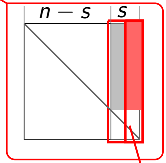
$$\mathcal{P} = \quad \mathcal{P}' \circ S'$$

(s, m)-linear
w.r.t. **standard basis**

Find only c vectors from V

First reveal (c, m) - linearity
then (s, m) - linearity

Transform input space



basis of V

(s, t)–linearity v.s. $\mathcal{MQ}$ crypto

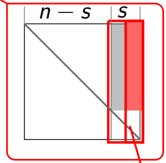
Type of **good key**

(s, m)-linear
w.r.t. **unknown** V

$$\mathcal{P} = \quad \mathcal{P}' \circ S'$$

(s, m)-linear
w.r.t. **standard basis**

Transform input space



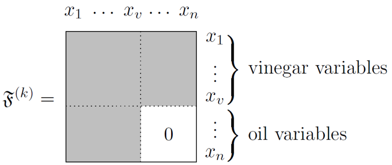
basis of V

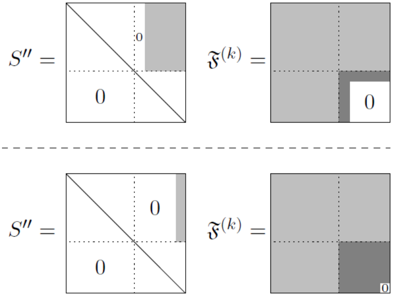
Find only c vectors from V

First reveal (c, m) - linearity
then (s, m) - linearity

UOV

$$f_s(x) = \sum_{i \in V, j \in V} \gamma_{ij}^{(s)} x_i x_j + \sum_{i \in V, j \in O} \gamma_{ij}^{(s)} x_i x_j,$$

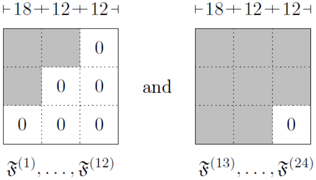


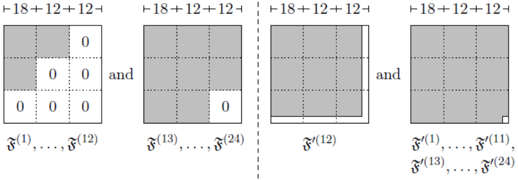


Good Keys for UOV

Combining strong (s,t)–linearity and (s,t)–linearity

Rainbow





We can choose $c_1 = c_2 = c = 1$

A challenge:

Post-Quantum Cryptography
from hard problems in Quasigroup theory

Canonical Identification Schemes

$\mathcal{P}, \mathsf{sk}$		$\mathcal{V}, \mathsf{pk}$
$\mathsf{com} \leftarrow_R \mathcal{P}_0(\mathsf{sk})$	$\xrightarrow{\mathsf{com}}$	
	$\xleftarrow{\mathsf{ch}}$	$\mathsf{ch} \leftarrow_R \mathsf{ChS}(1^k)$
$\mathsf{resp} \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch})$	$\xrightarrow{\mathsf{resp}}$	
		$b \leftarrow \mathsf{Vf}(\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})$

Informally:

1. Prover commits to some (randomized) value derived from sk
2. Verifier picks a challenge ‘ch’
3. Prover computes response ‘resp’
4. Verifier checks if response matches challenge

Properties of Canonical 3-pass IDS

$\mathcal{P}$, sk		$\mathcal{V}$, pk
com $\leftarrow_R \mathcal{P}_0$(sk)	com $\longrightarrow$	
	ch $\longleftarrow$	ch $\leftarrow_R$ ChS(1^k)
resp $\leftarrow \mathcal{P}_1$(sk, com, ch)	resp $\longrightarrow$	
		$b \leftarrow$ Vf(pk, com, ch, resp)

Soundness (with soundness error κ)

Probability that a PPT adversary $\mathcal{A}$ gets verified is $\kappa + \mathrm{negl}(k)$

- r rounds until $\kappa^r = negl(k)$
- guaranties negligible success of cheating Prover

Properties of Canonical 3-pass IDS

$\mathcal{P}, \mathsf{sk}$		$\mathcal{V}, \mathsf{pk}$
$\mathsf{com} \leftarrow_R \mathcal{P}_0(\mathsf{sk})$	$\xrightarrow{\mathsf{com}}$	
	$\xleftarrow{\mathsf{ch}}$	$\mathsf{ch} \leftarrow_R \mathsf{ChS}(1^k)$
$\mathsf{resp} \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch})$	$\xrightarrow{\mathsf{resp}}$	
		$b \leftarrow \mathsf{Vf}(\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})$

Special Soundness

There exists PPT algorithm $\mathcal{K}$ - knowledge extractor s.t. given two accepting transcripts:

$$\mathsf{trans} = (\mathsf{com}, \mathsf{ch}, \mathsf{resp}),\ \mathsf{trans}' = (\mathsf{com}, \mathsf{ch}', \mathsf{resp}'), \qquad \mathsf{ch} \neq ch',$$

extracts the secret sk with non-negligible probability

- random challenges can be answered only if Prover knows a witness
- implies soundness and knowledge

Properties of Canonical 3-pass IDS

$\mathcal{P}$, sk		$\mathcal{V}$, pk
com $\leftarrow_R \mathcal{P}_0(\mathsf{sk})$	$\xrightarrow{\text{com}}$	
	$\xleftarrow{\text{ch}}$	ch $\leftarrow_R \mathsf{ChS}(1^k)$
resp $\leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch})$	$\xrightarrow{\text{resp}}$	
		$b \leftarrow \mathsf{Vf}(\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})$

(statistical) Honest-Verifier Zero-Knowledge

There exists a PPT algorithm $\mathcal{S}$, called the simulator, such that the statistical distance between the real transcript and the simulated transcript is negligible in k.

- guaranties no leakage of secret

The Fiat-Shamir transform

IDS

$\mathcal{P}$		$\mathcal{V}$
$\mathsf{com} \leftarrow \mathcal{P}_0\ (\mathsf{sk})$	$\xrightarrow{\mathsf{com}}$	
	$\xleftarrow{\mathsf{ch}}$	$\mathsf{ch} \leftarrow_R \mathsf{ChS}\ (1^k)$
$\mathsf{resp} \leftarrow \mathcal{P}_1\ (\mathsf{sk}, \mathsf{com}, \mathsf{ch})$	$\xrightarrow{\mathsf{resp}}$	
		$b \leftarrow \mathsf{Vf}\ (\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})$

↓ ↓ ↓

FS signature

Signer
$\mathsf{com} \leftarrow \mathcal{P}_0(\mathsf{sk})$
$\mathsf{ch} \leftarrow H(m, \mathsf{com})$
$\mathsf{resp} \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch})$
output : $\sigma = (\mathsf{com}, \mathsf{resp})$

Verifier
$\mathsf{ch} \leftarrow H(m, \mathsf{com})$
$b \leftarrow \mathsf{Vf}(\mathsf{pk}, \mathsf{com}, \mathsf{ch}, \mathsf{resp})$
output : b

The Fiat-Shamir transform

IDS

$\mathcal{P}^r$		$\mathcal{V}^r$
$\text{com} \leftarrow \mathcal{P}_0{}^r(\text{sk})$	$\xrightarrow{\text{com}}$	
	$\xleftarrow{\text{ch}}$	$\text{ch} \leftarrow_R \text{ChS}^r(1^k)$
$\text{resp} \leftarrow \mathcal{P}_1{}^r(\text{sk}, \text{com}, \text{ch})$	$\xrightarrow{\text{resp}}$	
		$b \leftarrow \text{Vf}^r(\text{pk}, \text{com}, \text{ch}, \text{resp})$

$\downarrow \downarrow \downarrow$

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The Fiat-Shamir transform

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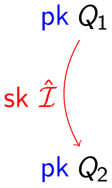
Provably secure IDS and signatures from quasigroups? Certainly!



$\mathcal{P}$, sk		$\mathcal{V}$, pk
$\hat{\mathcal{I}}_1 \leftarrow_R \mathcal{P}_0(\mathsf{sk})$	$\xrightarrow{\hat{\mathcal{I}}_1 \text{ (com)}}$	
	$\xleftarrow{\text{ch}}$	$\mathsf{ch} \leftarrow_R \{0,1\}$
$\hat{\mathcal{I}_{\mathsf{ch}}} \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch})$	$\xrightarrow{\hat{\mathcal{I}_{\mathsf{ch}}} \text{ (resp)}}$	
		$b \leftarrow \hat{\mathcal{I}_{\mathsf{ch}}}(Q_{\mathsf{ch}}) \stackrel{?}{=} \hat{Q}$

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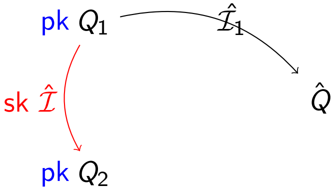

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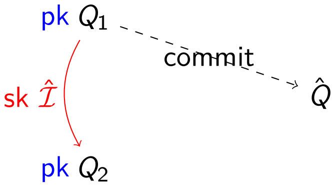

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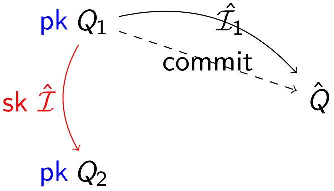

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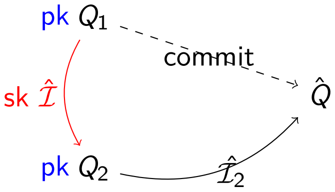

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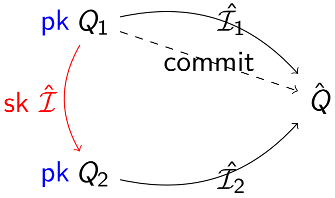

- Big communication cost
- Do we actually need quasigroups?
  - Works perfectly well for random isomorphic quadratic polynomials
  - no use of quasigroup structure

$\mathcal{P}$, sk — $\mathcal{V}$, pk

$\hat{\mathcal{I}}_1 \leftarrow_R \mathcal{P}_0(\mathsf{sk})$

$\xrightarrow{\hat{\mathcal{I}}_1 \text{ (com)}}$

$\xleftarrow{\mathsf{ch}}$ $\quad \mathsf{ch} \leftarrow_R \{0, 1\}$

$\hat{\mathcal{I}}_{\mathsf{ch}} \leftarrow \mathcal{P}_1(\mathsf{sk}, \mathsf{com}, \mathsf{ch})$

$\xrightarrow{\hat{\mathcal{I}}_{\mathsf{ch}} \text{ (resp)}}$

$b \leftarrow \hat{\mathcal{I}}_{\mathsf{ch}}(Q_{\mathsf{ch}}) \stackrel{?}{=} \hat{Q}$

Provably secure IDS and signatures from quasigroups?

- Can we use the problem of completing partial Latin squares?

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

$\longrightarrow$

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

- Hard problem even when only 3 integers remain unfilled
- Prover shows the knowledge of a “completion”
- Open problem: How to do it in Zero Knowledge manner
- Open problem: Evaluation of the practical security of the problem

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Thank you for listening!