BEYOND BARYCENTRIC ALGEBRAS AND CONVEX SETS

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OUTLINE

- Affine spaces, convex sets and barycentric algebras

- Extended barycentric algebras

- q-convex sets and q-barycentric algebras

- Threshold barycentric algebras and threshold affine spaces

AFFINE SUBSPACES of $\mathbb{R}^n$

$\mathbb{R}$ - the field of reals; $I^\circ :=]0,1[= (0,1) \subset \mathbb{R}$.

The **line** $L_{x,y}$ through $x, y \in \mathbb{R}^n$:
$L_{x,y} = \{xy\,\underline{p} = x(1-p) + yp \in \mathbb{R}^n \mid p \in \mathbb{R}\}$.

$A \subseteq \mathbb{R}^n$ is a (non-trivial) **affine subspace** of $\mathbb{R}^n$ if, together with any two distinct points x and y, it contains the line $L_{x,y}$.

One obtains an algebra $(A, \{\underline{p} \mid p \in \mathbb{R}\})$.

CONVEX SUBSETS of $\mathbb{R}^n$

The **line segment** $I_{x,y}$ joining the points x, y:
$I_{x,y} = \{xy\,\underline{p} = x(1-p) + yp \in \mathbb{R}^n \mid p \in I^\circ\}$.

$C \subseteq \mathbb{R}^n$ is a (non-trivial) **convex subset** of $\mathbb{R}^n$ if, together with any two distinct points x and y, it contains the line segment $I_{x,y}$.

One obtains an algebra $(C, \{\underline{p} \mid p \in I^\circ\})$.

AFFINE SPACES

F - a subfield of $\mathbb{R}$

An **affine space over** F (or **affine** F-**space**) - an algebra $(A, \underline{F})$, where

$$\underline{F} = \{\underline{p} \mid p \in F\}$$

and

$$xy\underline{p} = \underline{p}(x, y) = x(1 - p) + yp.$$

Note: $(A, \underline{F})$ is equivalent to the algebra

$$\left(A, \sum_{i=1}^{n} x_i r_i \,\middle|\, \sum_{i=1}^{n} r_i = 1 \text{ in } F\right).$$

THE VARIETY OF AFFINE F-SPACES

THEOREM: The class $\underline{\underline{F}}$ of all affine F-spaces is a variety (equationally-defined class).

It is axiomatized by the following:

idempotence: $xx\underline{p} = x$,

entropicity: $xy\underline{p}\ zt\underline{p}\ \underline{q} = xz\underline{q}\ yt\underline{q}\ \underline{p}$,

affine identities: $xy\underline{p}\ xy\underline{q}\ \underline{r} = xy\ \underline{pqr}$,

trivial identities: $xy\underline{0} = x = yx\underline{1}$,

for all $p, q, r \in F$.

For each $p \in F$ with $p \neq 0, 1$, the reduct $(A, \underline{p})$ of an affine F-space $(A, \underline{F})$ is an (idempotent and entropic) quasigroup.

Hence the algebra $(A, \underline{F} \setminus \{\underline{0}, \underline{1}\})$ is an (idempotent and entropic) **multi-quasigroup**.

BARYCENTRIC ALGEBRAS

F - a subfield of $\mathbb{R}$, $I^\circ :=]0,1[= (0,1) \subset F$.

A **convex set over** F (or **convex** F**-set**) - an algebra $(A, \underline{I}^o)$, where $\underline{I}^o = \{\underline{p} \mid p \in I^\circ\}$.

The class $\mathcal{C}$ of convex F-sets forms a quasivariety.
The variety generated by $\mathcal{C}$ is the variety $\mathcal{B}$ of **barycentric algebras**.

The variety $\mathcal{B}$ is axiomatized by the following:

idempotence (I): $\quad xx\underline{p} = x$,

skew-commutativity (SC):
$xy\,\underline{p} = xy\,\underline{1-p} =: xy\,\underline{p}'$,

skew-associativity (SA):
$[xy\underline{p}]\,z\,\underline{q} = x\,[yz\,\underline{q/(p\circ q)}\,]\,\underline{p\circ q}$

for all $p, q \in I^\circ$, where
$p \circ q = (p'q')' = p + q - pq$.

EXAMPLES

• **Convex subsets** of affine F-spaces under the operations

$$xy\,\underline{p} = xp' + yp = x(1-p) + yp$$

for each $p \in I^{\circ}$.

The subquasivariety $\mathcal{C}$ of the variety $\mathcal{B}$ is defined by the cancellation laws

$$(xy\,\underline{p} = xz\,\underline{p}) \to (y = z)$$

for all operations $\underline{p}$ of $\underline{I}^{\circ}$.

• **"Stammered" semilattices** $(S, \cdot)$ with the operation $x \cdot y = xy\underline{p}$ for all $p \in I^{\circ}$.
They form the subvariety $\mathcal{SL}$ of $\mathcal{B}$ defined by $xy\,\underline{p} = xy\,\underline{r}$ for all $p, r \in I^{\circ}$.

• Certain **sums of convex sets** over semilattices.

THEOREM: Each barycentric algebra is a subalgebra of a Płonka sum of convex sets over its semilattice replica.

EXTENDED BARYCENTRIC ALGEBRAS

Barycentric algebras may be considered as **extended barycentric algebras** $(A, \underline{I})$, where $I = [0, 1] \subset \mathbb{R}$, and with the operations $\underline{0}$ and $\underline{1}$ defined by

$$xy\,\underline{0} = x \text{ and } xy\,\underline{1} = y.$$

Skew associativity may also be written as:

$$[xy\underline{p}\,]z\underline{q} = x[\,yz\,\underline{(p \circ q \to q)}\,]\,\underline{p \circ q} \qquad \text{(SA)},$$

where

$$p \to q = \begin{cases} 1 & \text{if } p = 0; \\ q/p & \text{otherwise} \end{cases}$$

Proposition: The class $\overline{\mathcal{B}}$ of extended barycentric algebras is a variety, specified by the identities (I), (SC), (SA) and the two above.

EXTENDING THE CONCEPTS of a CONVEX SET and of a BARYCENTRIC ALGEBRA

Want to extend the concepts of a convex set and a barycentric algebra, while retaining as many key properties of barycentric algebras as possible.

Two types of extensions obtained by:

1. using different intervals of the field F;

2. using more general rings.

q-CONVEX SETS

A convex subset C of an affine F-space A is an $\underline{I}^{\circ}$-subreduct of A (subalgebra of $(A, \underline{I}^{\circ})$).

Replace the interval I° by an open interval $]q, q'[$, where $q \in F$ with $q \leq 1/2$ and $q' = 1 - q$.

A subalgebra $\left(C, \underline{]q, q'[}\right)$ of $\left(A, \underline{]q, q'[}\right)$ is called a **q-convex set**.

The class $\mathcal{C}_q$ of all q-convex subsets of affine F-spaces is a quasivariety. The variety $\mathcal{B}_q$ generated by the quasivariety $\mathcal{C}_q$ is called the variety of **q-barycentric algebras**.

Note: $\mathcal{C}_0 = \mathcal{C}$, and $\mathcal{B}_0 = \mathcal{B}$.
$\mathcal{B}_{1/2} = \mathcal{CBM}$ (the variety of commutative binary modes)

SOME BASIC PROPERTIES

Proposition: Let $t \in F$.
If $-\infty < t < 0$, then under the operations of $\underline{[t, t']}$ the line F is generated by $\{0, 1\}$.
If $0 < t < 1/2$, then under the operations of $\underline{[t, t']}$ the interval I is generated by $\{0, 1\}$.

Proposition: Free $\mathcal{B}_q$-algebra over X is isomorphic to the subalgebra generated by X in the $]q, q'[$-reduct $(XF, \underline{]q, q'[})$ of the free affine F-space $(XF, \underline{F})$ over X.

Corollary: In each q-convex set, the operations of $\underline{[t, t']}$, for $t \neq 0$ and $t \neq 1/2$, either generate all operations of $\underline{I}^{\circ}$, or all operations of $\underline{F}$.

CLASSIFICATION

THEOREM Let $q \in F$ with $q \leq 1/2$. Then each variety $\mathcal{B}_q$ is equivalent to one of the following:
(a) the variety $\mathcal{CBM}$ of commutative binary modes, if $q = 1/2$;
(b) the variety $\mathcal{B}$ of barycentric algebras, if $0 \leq q < 1/2$;
(c) the variety $\mathcal{A}$ of affine F-spaces, if $q < 0$.

THRESHOLD ALGEBRAS

Set a **threshold** t, where $t = -\infty$ or $t \in F$ with $t \leq 1/2$.

For elements x, y of an affine F-space, define

$$xy\,\underline{\underline{r}} = \begin{cases} x & \text{if } r < t; \\ xy\,\underline{r} = x(1-r) + yr & \text{if } t \leq r \leq t'; \\ y & \text{if } r > t' \end{cases}$$

for $r \in F$. Then the binary operations $\underline{\underline{r}}$ are described as **threshold-t affine combinations** (**small, moderate** and **large** respectively).

For a given threshold t, the algebra $(A, \underline{\underline{F}})$, where $\underline{\underline{F}} = \{\underline{\underline{r}} \mid r \in F\}$, is called a **threshold-t affine F-space**.

Proposition: Let t be a threshold. Let A be an affine F-space. Then under the threshold-t affine combinations $\underline{\underline{r}}$ for $r \in F$, the threshold-t affine F-space $(A, \underline{\underline{F}})$ is idempotent, entropic and skew-commutative.

For a given threshold t, the class $\mathcal{A}^t$ of **threshold-t affine F-spaces** is the variety generated by the class of affine F-spaces under the threshold-t affine combinations.

For $0 \leq t \leq 1/2$, similar definitions provide the concepts of
threshold-t convex combinations,
threshold-t convex sets, and the variety $\mathcal{B}^t$ of **threshold-t barycentric algebras**.

If $t = -\infty$, then $\mathcal{A} = \mathcal{A}^{-\infty}$.
If $t = 1/2$, then $\mathcal{A}^{1/2} \simeq \mathcal{B}^{1/2} \simeq \overline{\mathcal{CBM}}$,
If $0 < t < 1/2$, then $\mathcal{A}^t \simeq \mathcal{B}^t \simeq \overline{\mathcal{B}}$, $\mathcal{A}^0 \simeq \overline{\mathcal{B}}$.

MAIN RESULT

THEOREM Each variety of threshold affine F-spaces is equivalent to one of the following classes:
(a) the variety $\mathcal{A}$ of affine F-spaces;
(b) the variety $\overline{\mathcal{B}}$ of extended barycentric algebras;
(c) the variety $\overline{\mathcal{CBM}}$ of extended commutatve binary modes.

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