2-permutational left quasigroups

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Definition

A left quasigroup $(X, \circ, \backslash)$ is:

- 2-*permutational* if $(a \circ x) \circ y = (b \circ x) \circ y$, for all $a, b, x, y \in X$
- 2-*reductive* if $(a \circ x) \circ y = x \circ y$, for all $a, x, y \in X$

Example

2-permutational not 2-reductive: $(0 \circ 1) \circ 1 = 0 \circ 1 = 0 \neq 2 = 1 \circ 1$

$\circ$	0	1	2	3
0	1	0	3	2
1	3	2	1	0
2	1	0	3	2
3	3	2	1	0

- 2-permutational + idempotent $\Rightarrow$ 2-reductive.
- 2-permutational + left distributive (a rack) $\Rightarrow$ 2-reductive

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Right cyclic left quasigroups

Definition

A left quasigroup $(X, \circ, \backslash)$ is

- *right cyclic*, if $(x\backslash y)\backslash(x\backslash z) = (y\backslash x)\backslash(y\backslash z)$, for all $x, y, z \in X$
- *non-degenerate*, if the mapping $T\colon X \to X;\ x \mapsto x\backslash x$ is a bijection

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Right cyclic left quasigroup:

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Right cyclic left quasigroups

Theorem (W. Rump)

Each finite right cyclic left quasigroup is non-degenerate.

Example (W. Rump)

The quasigroup $(\mathbb{Z}, \circ, \backslash)$ with

$$x \circ y = y + min(x, 0) \quad \text{and} \quad x \backslash y = y - min(x, 0),$$

is right cyclic.

The mapping $f : \mathbb{Z} \to \mathbb{Z}$

$$x \mapsto x \backslash x = \begin{cases} x, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases}$$

is not a bijection.

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Example (2-permutational versus right cyclic)

2-permutational left quasigroup $(X, \circ, \backslash)$

$$\begin{array}{c|ccc} \circ & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 2 & 2 & 0 & 1 \end{array} \qquad \begin{array}{c|ccc} \backslash & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 2 & 0 \end{array},$$

but not right cyclic $0 = 0\backslash 1 = (0\backslash 1)\backslash 1 \neq (1\backslash 1)\backslash 1 = 2\backslash 1 = 2$.

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but not 2-permutational $2 = (0 \circ 1) \circ (0 \circ 0) \neq (1 \circ 0) \circ (1 \circ 0) = 0$.

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Medial right cyclic left quasigroups

Lemma

If $(X, \circ, \backslash)$ *is medial* ($(x \circ y) \circ (z \circ t) = (x \circ z) \circ (y \circ t)$)*, then*

$$\text{right cyclic} \quad \Longleftrightarrow \quad 2-\text{permutational}$$

Theorem (JPZ)

If $(X, \circ, \backslash)$ *is non-degenerate right cyclic then*

$$\text{medial} \quad \Longleftarrow \; \Longrightarrow \quad 2-\text{permutational}$$

Corollary

Each finite 2-permutational right cyclic left quasigroup is medial.

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Is every infinite 2-permutational right cyclic left quasigroup medial?

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YBE: $(id \times r)(r \times id)(id \times r) = (r \times id)(id \times r)(r \times id)$

Fact

Each (involutive) birack $(X, \circ, \bullet, \backslash, /)$ defines an (involutive) solution of YBE:

$$r(x, y) = (x \circ y, x \bullet y)$$

Each (involutive) solution $r = (\sigma, \tau)$ determines an (involutive) birack with

$$x \circ y = \sigma_x(y),\ x \bullet y = \tau_y(x),\ x\backslash y = \sigma_x^{-1}(y),\ x/y = \tau_y^{-1}(x)$$

Involutive birack: $x \bullet y = (x \circ y)\backslash x$

Theorem (W. Rump; P. Dehornoy)

An algebra $(X, \circ, \bullet, \backslash, /)$ is an involutive birack if and only if $(X, \circ, \backslash)$ is a non-degenerate right cyclic left quasigroup.

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Multipermutation involutive solutions of level 2

For an involutive birack $(X, \circ, \bullet, \backslash, /)$ the retraction relation:

$$x \sim y \;\Leftrightarrow\; x \circ z = y \circ z, \text{ for all } z \in X$$

is a congruence.
The quotient birack is denoted by $\mathrm{Ret}(X)$.

Proposition (T. Gateva-Ivanova)

Let $(X, \circ, \bullet, \backslash, /)$ be an involutive birack. Then $|\mathrm{Ret}(\mathrm{Ret}(X))| = 1$ if and only if the left quasigroup $(X, \circ, \backslash)$ is 2-permutational (2-permutational birack).

Fact

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2-reductive left quasigroups

Lemma

If $(X, \circ, \backslash)$ is left distributive, then the following are equivalent:

1. $(X, \circ, \backslash)$ *is right cyclic;*
2. $(X, \circ, \backslash)$ *is 2-reductive.*

Lemma

If $(X, \circ, \backslash)$ is 2-reductive, then the following are equivalent:

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The Structure Theorem for 2-reductive racks

Theorem (JPZ + D. Stanovský)

An algebra $(X, \circ, \backslash)$ *is a 2-reductive rack if and only if it is a disjoint union of abelian groups* A_j*,* $j \in I$*, with operations for* $x \in A_i$ *and* $y \in A_j$*:*

$$x \circ y = y + c_{i,j} \quad \text{and} \quad x \backslash y = x - c_{i,j},$$

where $A_j = \langle \{c_{i,j} \mid i \in I\} \rangle$*, for every* $j \in I$*.*

ALGORITHM: Outputs all 2-reductive racks of size n

1. For all partitionings $n = n_1 + n_2 + \cdots + n_k$ do (2–4).
2. For all abelian groups $A_1, \ldots, A_k$ of size $|A_i| = n_i$ do (3–4).
3. For all constants $a_{i,j} \in A_j$ do (4).
4. If, for all $1 \leq j \leq k$, we have $A_j = \langle \{a_{i,j} \mid i \in I\} \rangle$ then construct a rack as a sum of disjoint union of groups A_i.

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Enumeration

Theorem

Two 2-reductive racks $\mathcal{A} = ((A_i)_{i\in I};\ (c_{i,j})_{i,j\in I})$ and $\mathcal{B} = ((B_i)_{i\in I};\ (b_{i,j})_{i,j\in I})$, over the same index set I, are isomorphic if and only if there is a permutation $\pi \in S_n$ and group isomorphisms $\psi_i\colon A_i \to B_{\pi(i)}$ such that $\psi_j(c_{i,j}) = b_{\pi(i),\pi(j)}$.

The number of racks (P. Vojtěchovský and S.Y. Yang [2019]) and 2-reductive racks of size n, up to isomorphism

n	1	2	3	4	5	6	7	8	9	10
racks	1	2	6	19	74	353	2080	16023	159526	2093244
2-reductive	1	2	5	17	65	323	1960	15421	155889	2064688

n	11	12	13	14
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Example (2-reductive racks of size 3 and 4)

- One orbit: $((\mathbb{Z}_3),(1))$.
- Two orbits: $((\mathbb{Z}_2,\mathbb{Z}_1),\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right))$, $((\mathbb{Z}_2,\mathbb{Z}_1),\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right))$ and $((\mathbb{Z}_2,\mathbb{Z}_1),\left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right))$.
- Three orbits: $((\mathbb{Z}_1,\mathbb{Z}_1,\mathbb{Z}_1),\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix})$.

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- Four orbits: $((\mathbb{Z}_1,\mathbb{Z}_1,\mathbb{Z}_1,\mathbb{Z}_1),\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix})$.

Main Theorem

Theorem

Each 2-permutational right cyclic non-degenerate left quasigroup is isotopic to 2-reductive rack under isotopy (id,β,id), for some bijection β of the set X.

The proof is based on the following two facts:

- For a 2-reductive left quasigroup $(X, \circ, \backslash)$ and a bijection β such that

$$(1) \qquad \beta(y) \circ \beta(x \circ z) = \beta(x) \circ \beta(y \circ z) \text{ for all } x, y, z \in X,$$

  the left quasigroup $(X, *, \backslash_*)$ with $x * y = x \circ \beta(y)$ and $x \backslash_* y = \beta^{-1}(x \backslash y)$, is 2-permutational and right cyclic.

- For a 2-permutational medial left quasigroup $(X, \circ, \backslash)$ and $e \in X$, the left quasigroup $(X, *, \backslash_*)$ with

$$x * y = x \circ L_e^{-1}(y) = x \circ (e \backslash y) \quad \text{and} \quad x \backslash_* y = L_e(x \backslash y) = e \circ (x \backslash y),$$

  is a 2-reductive rack.
  The 2-reductive rack $(X, *, \backslash_*)$ is such that
  - $e * y = y$, for any $y \in X$;
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$$x * y = x \circ L_e^{-1}(y) = x \circ (e \backslash y) \quad \text{and} \quad x \backslash_* y = L_e(x \backslash y) = e \circ (x \backslash y),$$

  is a 2-reductive rack.
  The 2-reductive rack $(X, *, \backslash_*)$ is such that
  - $e * y = y$, for any $y \in X$;
  - the mapping $L_e \colon X \to X; x \mapsto e \circ x$, satisfies Condition (1).

How to obtain all 2-permutational right cyclic left quasigroups from 2-reductive racks

1. Take all 2-reductive racks $(X, \circ, \backslash)$ such that there exists $e \in X$ with $L_e = \mathrm{id}$
2. Take all permutations $\pi \in S_X$ which
   - satisfy Condition (1)

$$\pi(y) \circ \pi(x \circ z) = \pi(x) \circ \pi(y \circ z) \text{ for all } x, y, z \in X,$$

   - are such that $(x \circ \pi(y)) \circ z \neq y \circ z$ for some $x, y, z \in X$
3. Construct the left quasigroup $(X, *, \backslash_*)$, with

$$x * y = x \circ \pi(y) \quad \text{and} \quad x \backslash_* y = \pi^{-1}(x \backslash y)$$

Example (1)

The 2-permutational right cyclic left quasigroup

$\circ$	0	1	2	3	4
0	0	2	1	4	3
1	3	2	1	0	4
2	4	2	1	3	0
3	0	2	1	4	3
4	0	2	1	4	3

can be obtain from two non-isomorphic 2-reductive racks:

$*_1$	0	1	2	3	4
0	0	1	2	3	4
1	3	1	2	4	0
2	4	1	2	0	3
3	0	1	2	3	4
4	0	1	2	3	4

and

$*_2$	0	1	2	3	4
0	4	1	2	0	3
1	0	1	2	3	4
2	3	1	2	4	0
3	4	1	2	0	3
4	4	1	2	0	3

Example (2)

The 2-reductive rack

$*$	0	1	2	3
0	0	1	2	3
1	2	3	0	1
2	0	1	2	3
3	2	3	0	1

can be isotopic to two non-isomorphic 2-permutational right cyclic left quasigroups:

$\circ_1$	0	1	2	3
0	1	0	3	2
1	3	2	1	0
2	1	0	3	2
3	3	2	1	0

$\circ_2$	0	1	2	3
0	1	2	3	0
1	3	0	1	2
2	1	2	3	0
3	3	0	1	2

The number of right cyclic 2-permutational left quasigroups of size n

n	1	2	3	4	5	6	7	8
right cyclic l.q.	1	2	5	23	88	595	3456	34528
2-permutational right cyclic l.q.	1	2	5	19	70	359	2095	16332
2-reductive racks	1	2	5	17	65	323	1960	15421
2-permutational, not 2-reductive	0	0	0	2	5	36	135	911

There are 23 right cyclic 2-permutational left quasigroups of size 4, up to isomorphism

- 17 are 2-reductive
- Two are 2-permutational

$\circ_1$	0	1	2	3
0	1	0	3	2
1	3	2	1	0
2	1	0	3	2
3	3	2	1	0

$\circ_2$	0	1	2	3
0	1	2	3	0
1	3	0	1	2
2	1	2	3	0
3	3	0	1	2

- Two are 3-permutational $((a \circ x) \circ y) \circ z = ((b \circ x) \circ y) \circ z$

$\circ_3$	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	3	2
3	1	0	3	2

$\circ_4$	0	1	2	3
0	1	0	2	3
1	1	0	2	3
2	0	1	3	2
3	1	0	3	2

- Two are not k-permutational for any $k \in \mathbb{N}$