Functional equations and their graphs

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Dedicated

to our coauthor

Slobodan K. Simić

1948–2019

Quadratic equations

We consider equations which are:

- Functional (there is at least one functional variable)
- On quasigroups (functional variables are interpreted as quasigroups)
- Generalized (each functional variable have single occurrence)
- Quadratic (object variables have exactly two occurrences).

Balanced (permutational) equations

There are also two important special cases:

- Balanced (permutational) equations
  (object variables have exactly two occurrences, once on each side of the equality sign)
- Balanced equations of the first kind
  (the order of occurrence of object variables is the same on both sides of the equality sign).

The most important paper in the field:

J. Aczél, V. D. Belousov, M. Hosszú:
Generalized associativity and bisymmetry on quasigroups,
Acta Math. Acad. Sci. Hungar. **11** (1960), 127–136.

The following result is proved:

Generalized associativity and bisymmetry

Theorem

All quasigroups satisfying generalized associativity:

$$A(B(x,y),z) = E(x,F(y,z)),$$

(resp. generalized bisymmetry:

$$A(B(x,y),C(u,v)) = E(F(x,u),G(y,v)))$$

are isotopic to the same group (resp. Abelian group).

A. Sade:
Entropie demosienne de multigoupoides et de quasigroupes,
Ann. Soc. Scient. Bruxelles 73/3 (1959), 302–309.

V. D. Belousov:
Balanced identities in quasigroups (Russian),
Matem. Sb. t. 70 (112), no. 1 (1966), 55–97.

B. Alimpić:
Balanced laws on quasigroups (Serbian),
Mat. Vesnik 9(24) (1972), 249–255.

Quadratic equations

Quadratic equations were introduced in:

A. Krapež:
Strictly quadratic functional equations on quasigroups I,
Publ. Inst, Math. (Beograd) (N.S.) 29 (43), (1981), 125–138.

Above paper also gives a general solution
when all operations from the equation are mutually isostrophic
(see below).

Quadratic equations

Definition

Let Φ be the set of all operational symbols of a qudratic equation Eq and let $F, G \in \Phi$.

- $F{\sim}G$ iff Eq implies isostrophy of F and G.
- $F \approx G$ ($F \leftrightarrow G$) iff this isostrophy is even (odd)
  ($F \approx G$ ($F \leftrightarrow G$) iff F^α is isotopic to G
  and α is an even (odd) permutation of S_3).

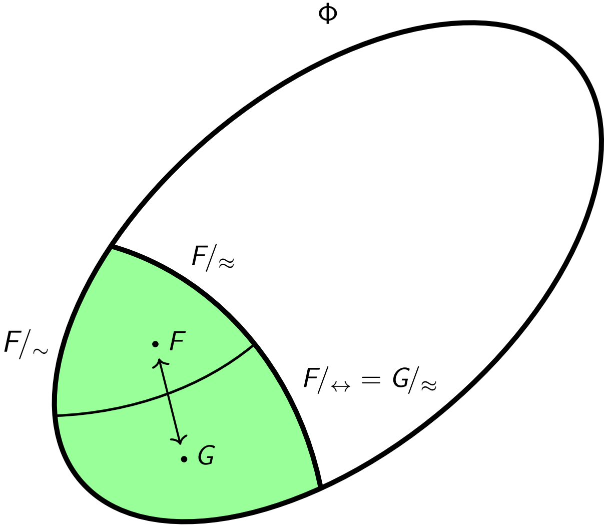
Quadratic equations

Lemma

Relations $\sim$ *and* $\approx$ *are equivalence relations.*
Define $F/_{\leftrightarrow} = \{G \in \Phi | F \leftrightarrow G\}$. *Then:*

- $F/_{\sim} = F/_{\approx} \cup F/_{\leftrightarrow}$
- *Either* $F/_{\approx} \cap F/_{\leftrightarrow} = \phi$ *or* $F/_{\approx} = F/_{\leftrightarrow}$.

Quadratic equations



Also possible:

$F/_{\leftrightarrow}$ is empty

$F/_{\leftrightarrow} = F/_{\approx}$

In his PhD thesis S. Krstić proved:

Theorem

All quasigroups from $F/_{\sim}$ are isostrophic to the same loop $\bullet$.

If $|F/_{\sim}| > 2$ then $\bullet$ is a group.

If $F/_{\approx} = F/_{\leftrightarrow}$ then $\bullet$ is an Abelian group.

Loops pertaining to $\sim$-classes have a common unit, but otherwise are independent of each other.

Theorem

Generalized quadratic quasigroup functional equation Eq is parastrophically uncancellable (i.e. $\sim = \square$) iff Krstić graph $K(Eq)$ is 3–connected.

Also:

Theorem

Quadratic quasigroup Eq *and* Eq$'$ *are parastrophically equivalent iff*
their Krstić graphs K(Eq) *and* K(Eq$'$) *are isomorphic.*

Definitions

Parastrophic equivalence is defined by example:

Functional equations of generalized associativity:

$$A(B(x, y), z) = E(x, F(y, z))$$

and generalized transitivity:

$$A(B(x, y), C(y, u)) = E(x, u)$$

are parastrophically equivalent because $F = C^{-1}$
i.e. they are parastrophes of each other.

Definitions

Krstić graphs are finite nonempty connected cubic multigraphs.

Definitions

For a generalized quadratic equation Eq,
the Krstić graph K(Eq) is given by:

- The vertices of K(Eq) are operation symbols from Eq
- The edges of K(Eq) are subterms of Eq
- If $F(s, t)$ is a subterm of Eq
  then the vertex F is incident to edges $s, t, F(s, t)$
  and no others.

Definition

$F \equiv G$ iff F and G are 3-connected in $K(Eq)$.

Theorem

Let Eq be a generalized quadratic quasigroup functional equation with operation symbols F and G.

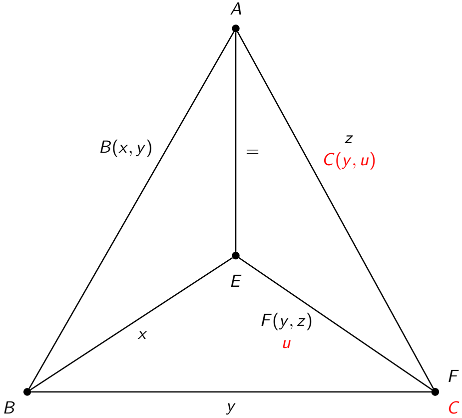
$F \sim G$ *in Eq iff* $F \equiv G$ *in* $K(Eq)$.

$|F/_{\sim}| > 2$ *(group case)*
iff tetrahedron K_4
is homeomorphically embedable in $K(Eq)$ *within* $F/_{\equiv}$.

$F/_{\approx} = F/_{\leftrightarrow}$ *(Abelian group case)*
iff subgraph $F/_{\equiv}$ *is not planar*
iff the complete bipartite graph $K_{3,3}$
is homeomorphically embeddable in $K(Eq)$ *within* $F/_{\equiv}$.

Example

$A(B(x,y),z) = E(x,F(y,z))$ $\qquad$ $A(B(x,y),C(y,u)) = E(x,u)$



Example

There are exactly 100 generalized quadratic equations parastrophically equivalent to generalized associativity.

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We need to augment properties of Krstić graphs
so that such graph uniquely defines corresponding equation.

Eq-graphs

An Eq-*graph* is a structure $\Gamma = (V, E; I, \boldsymbol{d}, \chi, \omega)$ where:

(1) the triple $G = (V, E; I)$
is an underlying Krstić (multi)graph of Γ;

(2) $\boldsymbol{d} \in E$ is the unique designated edge;

(3) $\chi : V \to \{\mathit{red}, \mathit{blue}\}$ is a vertex (bi)coloring;

(4) ω is a bidirection of edges defined below.

Bidirection

Bidirection is a mapping $\omega : uv \mapsto \{(u, \lambda), (v, \mu)\}$ where $\lambda, \mu \in \{0, 1, 2\}$. The numbers correspond to direction of edges at each end: the "incoming" direction (0), and two "outcomming" directions (1,2).

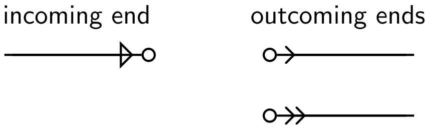


Figure: Incoming and outcoming ends of edges.

Bidirection

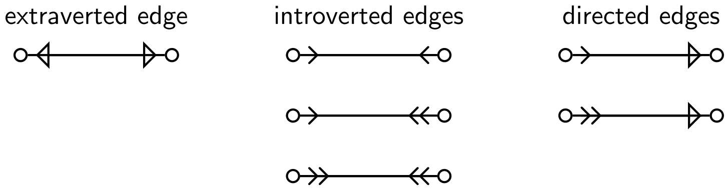


Figure: Different types of bidirected edges.

Bidirection

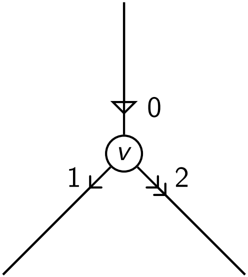
Bidirected graphs were defined first in:

J. Edmonds, E. L. Johnson:
Matching: A Well–Solved Class of Integer Linear Programs,
in the book:
M. Junger, G. Reinelt, G. Rinaldi (eds.):
Combinatorial Optimization Eureka, You Shrink!,
Lecture Notes in Computer Science 2570,
Springer
Berlin, Heidelberg
(2003)

but we have edges with two different types of outcomming ends.

Bidirection

With such definition, at every vertex we have situation like this:

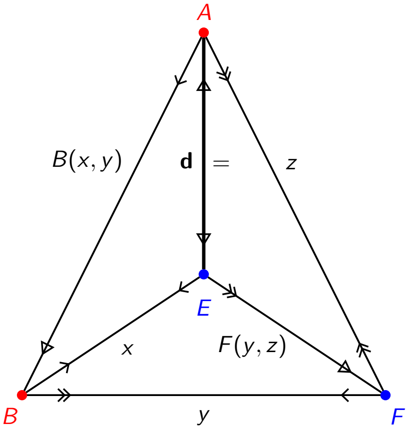


$$\deg(v) = 3$$
$$\mathrm{indeg}(v) = 1$$
$$\mathrm{outdeg}(v) = 2$$

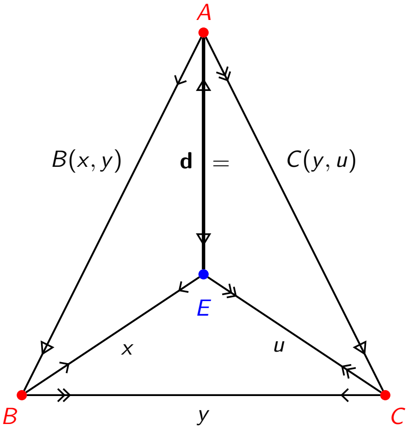
Figure: The degrees of a vertex.

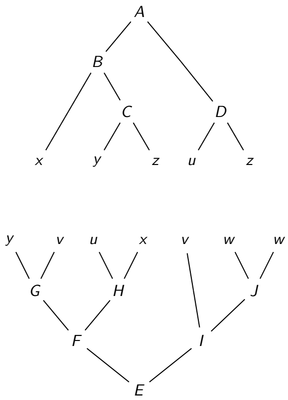
Example

$$A(B(x,y),z) = E(x,F(y,z))$$

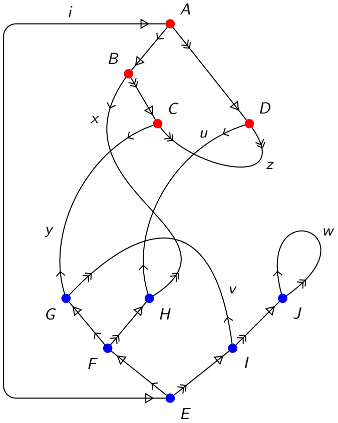


$$A(B(x,y),C(y,u)) = E(x,y)$$





(a)



(b)

(a) The trees of terms s and t of the equation $s = t$
(b) The graph $\Gamma(s = t)$ of the equation $s = t$

Axioms

The structure $\Gamma = (V, E; I, \mathbf{d}, \chi, \omega)$ is an *Eq*-graph if:

1. The multigraph $(V, E; I)$ is a Krstić graph (finite nonempty connected cubic graph);
2. Every vertex has indegree 1 and outdegree 2;
3. All loops are either directed or introverted;
4. The designated edge **d** is either extraverted or directed;
5. An extraverted edge, if it exists, is equal to **d**;
6. If **d** is directed then Γ is monochromatic;
7. If **d** is extraverted then the end vertices of **d** are of different colour;
8. End vertices of a directed edge are of the same colour.

Result

Theorem

Generalized quadratic quasigroup functional equations Eq *and* Eq$'$ *are logically equivalent*
iff
their Eq-graphs $\Gamma(\mathrm{Eq})$ *and* $\Gamma(\mathrm{Eq}')$ *are isomorphic.*