Biquandle cocycle invariants of surface-links

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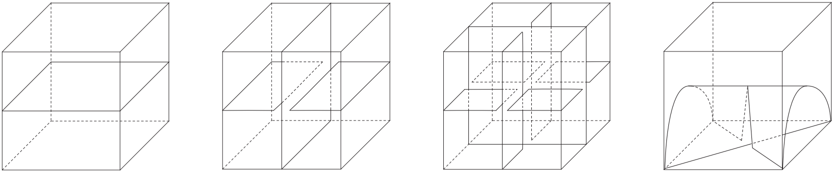
Broken surface diagrams

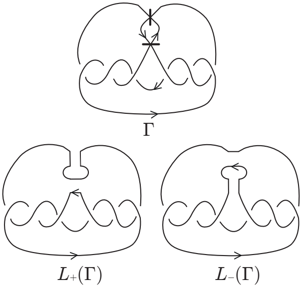
- A surface-link is a closed surface smoothly embedded in $\mathbb{R}^4$.
- If a surface-link is oriented, then we call it an oriented surface-link.
- A broken surface diagram of a surface-link $\mathscr{L}$ in $\mathbb{R}^4$ is a generic surface of $\mathscr{L}$ into $\mathbb{R}^3$ with over/under sheet information at each double curve.



(a) (b) (c) (d)

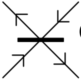
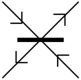
Marked graph diagrams

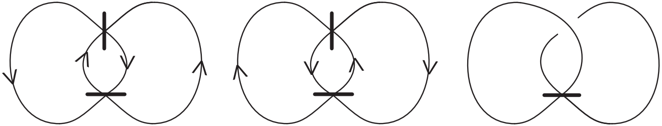
- A marked graph is a finite spatial regular graph with 4-valent rigid vertices such that each vertex has a marker.
- A diagram of a marked graph in $\mathbb{R}^2$ is called a marked graph diagram or ch-diagram.



- A marked graph diagram is said to be admissible if both resolutions $L_+(\Gamma)$ and $L_-(\Gamma)$ are diagrams of trivial links.

An oriented marked graph diagram of an oriented surface-link

- An orientation of a marked graph G in $\mathbb{R}^3$ is a choice of an orientation for each edge of G in such a way that every rigid vertex in G looks like  or , up to rotation.
- A marked graph in $\mathbb{R}^3$ is said to be orientable if it admits an orientation. Otherwise, it is said to be unorientable.



Theorem (Kawauchi-Shibuya-Suzuki, Yoshikawa)

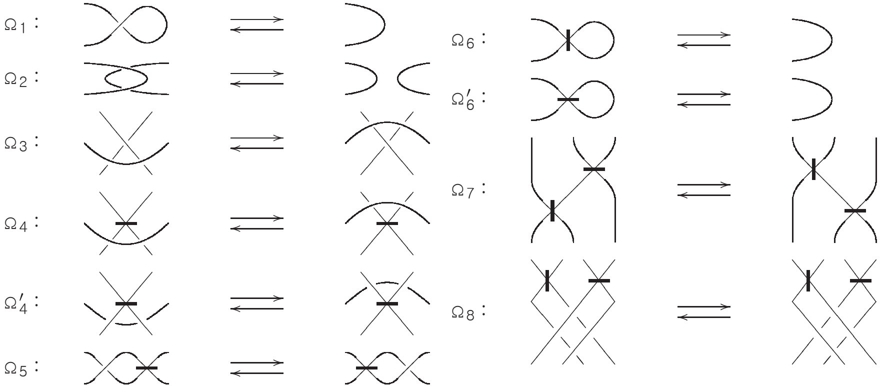
(1) Let $\mathscr{L}$ be a surface-link. Then there is an admissible marked graph diagram Γ s.t. $\mathscr{L}$ is presented by Γ.

(2) Let Γ be an admissible marked graph diagram. Then there is a surface-link $\mathscr{L}$ s.t. $\mathscr{L}$ is presented by Γ.

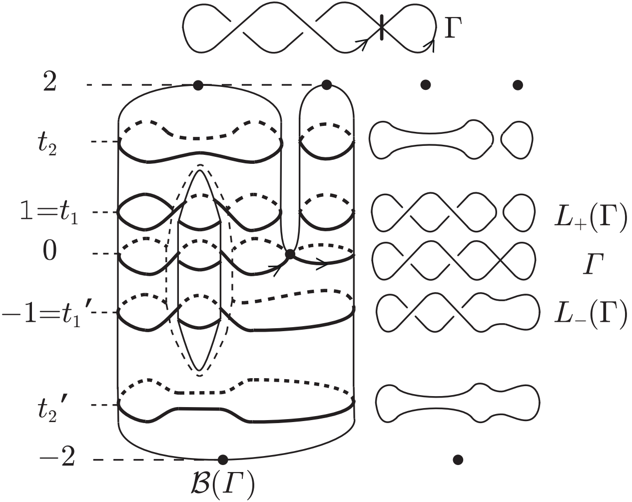
Theorem (Kearton-Kurlin, Swenton)

Two marked graph diagrams represent the same surface-link if and only if they are transformed into each other by a finite sequence of Yoshikawa moves.

Yoshikawa moves



Broken surface diagrams associated to marked graph diagrams



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Biquandles

Definition

A biquandle X is a set with two binary operations $\underline{\triangleright}, \overline{\triangleright} : X \times X \to X$ such that

(1) For any $x \in X$, $x \underline{\triangleright} x = x \overline{\triangleright} x$.

(2) Two binary operations $\underline{\triangleright}, \overline{\triangleright}$ are right invertible.

(3) The map $H : X \times X \to X \times X$ defined by $(x, y) \mapsto (y \overline{\triangleright} x, x \underline{\triangleright} y)$ is invertible.

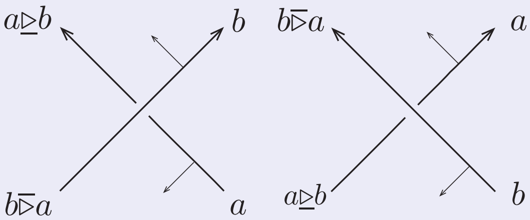
(4) For any $x, y, z \in X$,

$$(x \underline{\triangleright} y) \underline{\triangleright} (z \underline{\triangleright} y) = (x \underline{\triangleright} z) \underline{\triangleright} (y \overline{\triangleright} z),$$
$$(x \underline{\triangleright} y) \overline{\triangleright} (z \underline{\triangleright} y) = (x \overline{\triangleright} z) \underline{\triangleright} (y \overline{\triangleright} z),$$
$$(x \overline{\triangleright} y) \overline{\triangleright} (z \overline{\triangleright} y) = (x \overline{\triangleright} z) \overline{\triangleright} (y \underline{\triangleright} z).$$

Biquandle colorings of link diagrams

Definition

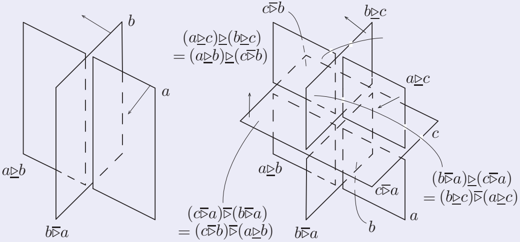
Let X be a biquandle. A (biquandle) coloring on an oriented link diagram is a function $\mathscr{C} : S \to X$, where S is the set of semi-arcs in the diagram, satisfying the condition depicted in the below figures.



Biquandle colorings of broken surface diagrams

Definition

A (biquandle) coloring on an oriented broken surface diagram is a function $\mathscr{C} : S \to X$, where S is the set of semi-sheets, satisfying the following condition at the double point set.



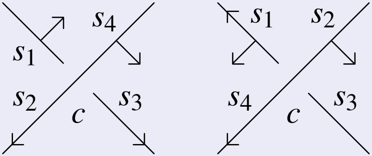
Biquandle coloring of marked graph diagrams

Definition

Let Γ be an oriented marked graph diagram and X a finite biquandle. A coloring of Γ is $\mathscr{C} : S(\Gamma) \to X$, where $S(\Gamma)$ is the set of semi-arcs in Γ, satisfying the following conditions:

(1) For each crossing $c \in C(\Gamma)$,

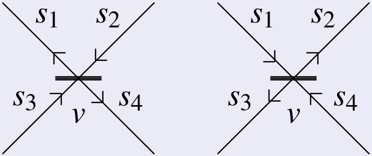
$$\mathscr{C}(s_3) = \mathscr{C}(s_1) \underline{\triangleright} \mathscr{C}(s_2), \; \mathscr{C}(s_4) = \mathscr{C}(s_2) \overline{\triangleright} \mathscr{C}(s_1).$$



Definition (continued)

(2) For each marked vertex $v \in V(\Gamma)$,

$$\mathscr{C}(s_1) = \mathscr{C}(s_2) = \mathscr{C}(s_3) = \mathscr{C}(s_4).$$



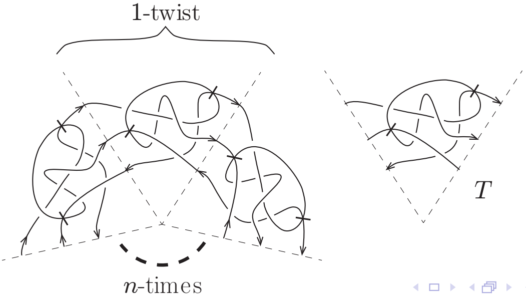
We denote by $\mathrm{Col}_X(\Gamma)$ the set of colorings of Γ.

Example

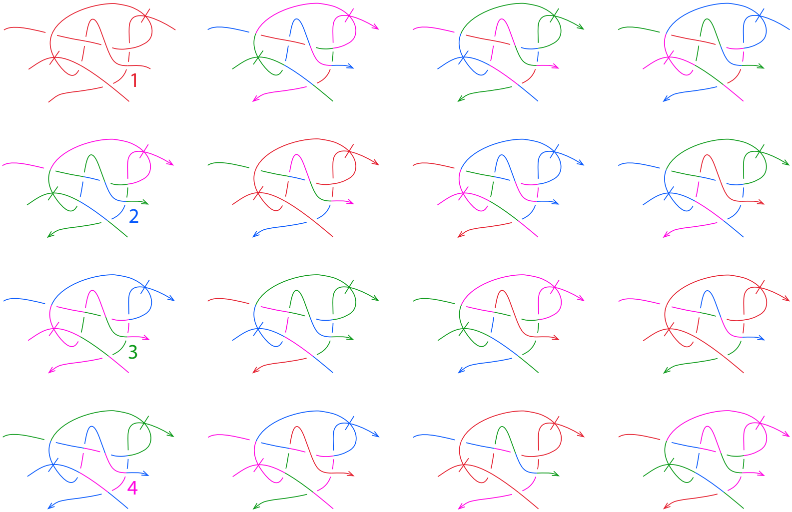
Let

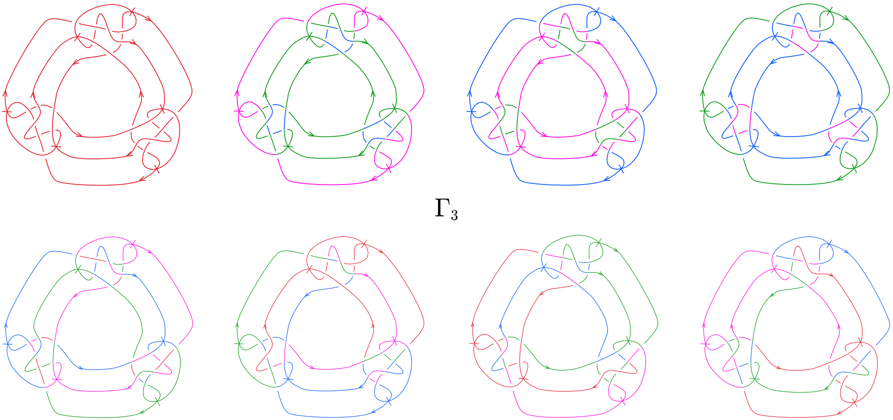
$$M = [\{m^1_{i,j}\}_{1\le i,j\le 4} | \{m^2_{i,j}\}_{1\le i,j\le 4}] = \left[\begin{array}{cccc|cccc} 1 & 4 & 2 & 3 & 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 4 & 3 & 3 & 3 & 3 \\ 3 & 2 & 4 & 1 & 4 & 4 & 4 & 4 \\ 4 & 1 & 3 & 2 & 2 & 2 & 2 & 2 \end{array}\right],$$

and $X = \{1,2,3,4\}$ the biquandle, where $i \underline{\triangleright} j = m^1_{i,j}$ and $i \overline{\triangleright} j = m^2_{i,j}$.
Let Γ_n be a marked graph diagram of n twist spun trefoil knot.



Colorings of T





Γ_3

$\#\mathrm{Col}_X(\Gamma_{3k-2}) = \#\mathrm{Col}_X(\Gamma_{3k-1}) = 4$, $\#\mathrm{Col}_X(\Gamma_{3k}) = 4 + (4 \times 3) = 16$ for $k \geq 1$.

Biquandle cocycles

Let X be a finite biquandle and A an abelian group with the identity element 1. Carter-Elhamdadi-Saito defined biquandle homology group $H_*^Q(X;A)$ and the biquandle cohomology group $H_Q^*(X;A)$.

Note that a biquandle 2-cocycle $f : C_2^Q(X) \to A$ satisfies

(1) $f(x,x) = 1$ for all $x,y \in X$.

(2) $f(y,z)f(x,y)f(x \underline{\triangleright} y, z \overline{\triangleright} y) = f(x,z)f(y \overline{\triangleright} x, z \overline{\triangleright} x)f(x \underline{\triangleright} z, y \underline{\triangleright} z)$, for each $x,y,z \in X$.

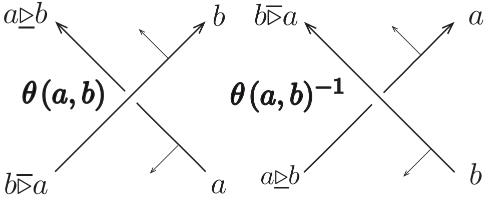
Note that a biquandle 3-cocycle $f : C_3^Q(X) \to A$ satisfies

(1) $f(x,x,y) = 1$ and $f(x,y,y) = 1$ for all $x,y \in X$.

(2) $f(y,z,w)f(x,y,w)f(x \underline{\triangleright} y, z \overline{\triangleright} y, w \overline{\triangleright} y)f(x \underline{\triangleright} w, y \underline{\triangleright} w, z \underline{\triangleright} w)$
$= f(x,z,w)f(x,y,z)f(y \overline{\triangleright} x, z \overline{\triangleright} x, w \overline{\triangleright} x)f(x \underline{\triangleright} z, y \underline{\triangleright} z, w \overline{\triangleright} z)$, for each $x,y,z,w \in X$.

Biquandle cocycle invariants of links

Let D be an oriented diagram of a link L and a coloring $\mathscr{C}$ of D given. Let $\theta \in Z^2_Q(X;A)$.



The partition function of D is defined by

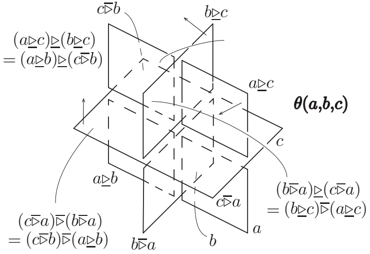
$$\Phi_\theta(D) = \sum_{\mathscr{C}} \prod_{c} B_\theta(c, \mathscr{C}) \in \mathbb{Z}[A].$$

Theorem (Carter-Elhamdadi-Saito)

Let L be a link and D a diagram of L. Then the partition function $\Phi_\theta(D)$ is an invariant of L, which is called the biquandle cocycle invariant of L and denoted by $\Phi_\theta(L)$.

Biquandle cocycle invariants of surface-links

Let $\mathscr{B}$ be an oriented diagram of a surface-link $\mathscr{L}$ and a coloring $\mathscr{C}$ of $\mathscr{B}$ given. Let $\theta \in Z^3_Q(X;A)$.



The partition function of $\mathscr{B}$ is defined by

$$\Phi_\theta(\mathscr{B}) = \sum_{\mathscr{C}} \prod_{\tau} B_\theta(\tau, \mathscr{C}) \in \mathbb{Z}[A].$$

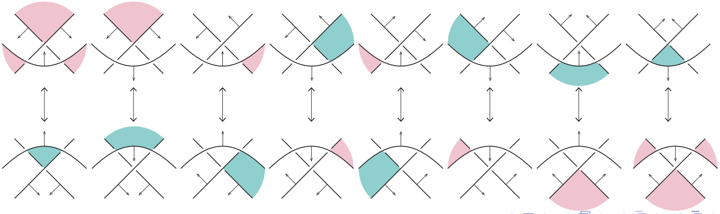
Theorem

Let $\mathscr{L}$ be a surface-link and $\mathscr{B}$ a diagram of $\mathscr{L}$. Then the partition function $\Phi_\theta(\mathscr{B})$ is an invariant of $\mathscr{L}$, which is called the biquandle cocycle invariant of $\mathscr{L}$ and denoted by $\Phi_\theta(\mathscr{L})$.

Biquandle cocycle invariants via mgd

Let D be a marked graph diagram of a surface-link.
There are two sequences $D_1 = L_+(D) \to \cdots \to D_m = O^r$,
$D'_1 = L_-(D) \to \cdots \to D'_n = O^s$.
Define $I^3_+ = \{i | D_i \to D_{i+1}$ is a Reidemeister move $3\}$ and
$I^3_- = \{j | D'_j \to D'_{j+1}$ is a Reidemeister move $3\}$.

Let $i \in I^3_+$. (resp., $j \in I^3_-$.) Exactly one of D_i and D_{i+1} (resp., D'_j and D'_{i+1}) has the region from which all normal orientations point outward such that the number of intersecting semi-arcs is 3. Let the region call the source region of i (resp., j).



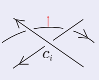
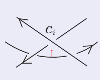
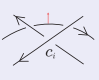
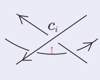
Definition

Let $\mathscr{L}$ be an oriented surface-link and Γ a marked graph diagram of $\mathscr{L}$. Let $\mathscr{C} : S(\Gamma) \to X$ be a coloring of Γ and $\theta \in Z^3_Q(X;A)$.

(1) Let $i \in I^3_+$. The **(Boltzman) weight** $B_\theta(i,\mathscr{C})$, for $i \in I^3_+$, is defined by

$$B_\theta(i,\mathscr{C}) = \theta(x_1,x_2,x_3)^{\varepsilon_{tm}(i)\varepsilon_b(i)},$$

where x_1, x_2 and x_3 are colors of the bottom, middle and top arcs, respectively, those bound the source region of i.

	$\varepsilon_b(i)=1$	$\varepsilon_b(i)=-1$
$\varepsilon_{tm}(i)$ $=1$		
$\varepsilon_{tm}(i)$ $=-1$		

Definition (continued)

(2) Let $j \in I^3_-$. The (Boltzman) weight $B_\theta(j, \mathscr{C})$, for $j \in I^3_-$, is defined by

$$B_\theta(j, \mathscr{C}) = \theta(x_1, x_2, x_3)^{-\varepsilon_{tm}(j)\varepsilon_b(j)},$$

where x_1, x_2 and x_3 are colors of the bottom, middle and top arcs, respectively, those bound the source region of j.

Definition

Let Γ be a marked graph diagram of an oriented surface-link $\mathscr{L}$. The partition function or state-sum (associated with θ) of a marked graph diagram Γ is defined by the state-sum expression

$$\Phi_\theta(\Gamma) = \sum_{\mathscr{C} \in \mathrm{Col}_X(\Gamma)} \prod_{x \in I^3_+ \cup I^3_-} B_\theta(x, \mathscr{C}),$$

where $B_\theta(x, \mathscr{C})$ is a weight of $x \in I^3_+ \cup I^3_-$.

Theorem (Kamada-Kawauchi-K.-Lee)

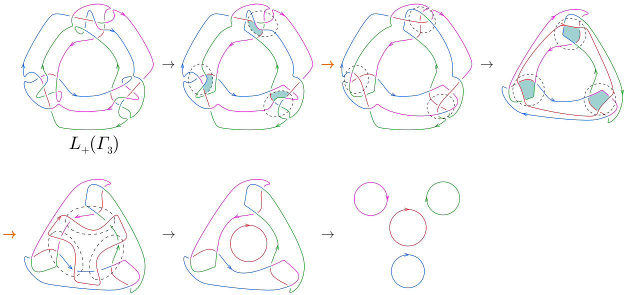
Let $\mathscr{L}$ be an oriented surface-link and Γ a marked graph diagram of $\mathscr{L}$. Then for any $\theta \in Z^3_Q(X;A)$, $\Phi_\theta(\mathscr{L}) = \Phi_\theta(\Gamma)$.

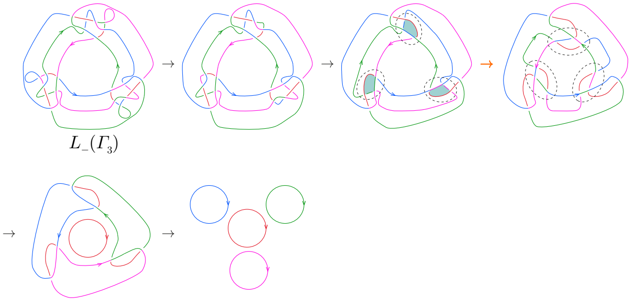
Example

Let

$$X = \left[\begin{array}{cccc|cccc} 1 & 4 & 2 & 3 & 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 4 & 3 & 3 & 3 & 3 \\ 3 & 2 & 4 & 1 & 4 & 4 & 4 & 4 \\ 4 & 1 & 3 & 2 & 2 & 2 & 2 & 2 \end{array}\right],$$

and $\theta = \chi_{(1,4,1)}\chi_{(1,4,3)}\chi_{(2,4,1)}\chi_{(2,4,3)}\chi_{(3,2,1)}\chi_{(3,2,3)}\chi_{(4,2,1)}\chi_{(4,2,3)}$ a cocycle with the coefficient $\mathbb{Z}_2 =< t | t^2 = 1 >$, where $\chi_{(a,b,c)}(x,y,z)$ is defined to be t if $(x,y,z) = (a,b,c)$ and 1 otherwise.





Then $\prod_{x\in I_+^3\cup I_-^3} B_\theta(x,\mathscr{C}) = \theta(1,1,4)\theta(1,1,3)\theta(1,1,2)\theta(1,2,1)$ $\theta(1,4,1)\theta(1,3,1)\theta(2,1,2)^{-1}\theta(4,1,4)^{-1}\theta(3,1,3)^{-1} = t$, where $\theta = \chi_{(1,4,1)}\chi_{(1,4,3)}\chi_{(2,4,1)}\chi_{(2,4,3)}\chi_{(3,2,1)}\chi_{(3,2,3)}\chi_{(4,2,1)}\chi_{(4,2,3)}$.

The biquandle cocycle invariant is

$$\Phi_\theta(\Gamma) = \sum_{\mathscr{C}\in \mathrm{Col}_X(\Gamma)} \prod_{x\in I_+^3\cup I_-^3} B_\theta(x,\mathscr{C}) = 4+12t.$$

Thank you