On relationship between a 2nd group cohomology group and a 2nd quandle cohomology group

Byeorhi Kim
(a joint work with Yongju Bae and J. Scott Carter)

School of Mathematics
Kyungpook National University

July. 08. 2019

Definition

A quandle is a set Q equipped with a binary operation $\lhd : Q \times Q \longrightarrow Q$ satisfying the following axioms ;

1. *For all $a \in Q$, $a \lhd a = a$,*
2. *For all $a, b \in Q$, $\exists! c \in Q$ such that $c \lhd a = b$,*
3. *For all $a, b, c \in Q$, $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$.*

Example

T_3	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

$Q(4,1)$	0	1	2	3
0	0	3	1	2
1	2	1	3	0
2	3	0	2	1
3	1	2	0	3

- For a set Q, define $\triangleleft : Q \times Q \to Q$ by $x \triangleleft y = x$, then it is a quandle, called the *trivial quandle*. In this talk, we denote the trivial quandle with n elements by T_n.
- The *tetrahedral quandle* $Q(4,1)$ is the quandle of order four with the operation $\diamond : Q(4,1) \times Q(4,1) \to Q(4,1)$ defined by the above table. Here, $Q(n,i)$ is the i-th connected quandle of order n in Vendramin's list.

  L. Vendramin, On the classification of quandles of low order, J. Knot Theory Ramifications, **21(9)** *(2012) 125008, 10pp.*

Definition

Let $(Q, \triangleleft)$ and $(P, \diamond)$ be quandles. Let $f : Q \to P$ be a function.

- f is called a *homomorphism* if $f(x \triangleleft y) = f(x) \diamond f(y)$ for all $x, y \in Q$.
- A bijective quandle homomorphism is called a *isomorphism*.
- An *automorphism* is a quandle isomorphism from a quandle to the quandle itself.
- The *automorphism group* of $(Q, \triangleleft)$ is the group of all automorphisms on $(Q, \triangleleft)$, denoted by $\mathrm{Aut}(Q)$.
- Then *inner automorphism group*, denoted by $\mathrm{Inn}(Q)$, is the subgroup of $\mathrm{Aut}(Q)$ which is generated by $\{(- \triangleleft x), (- \bar{\triangleleft}\, x) \mid x \in Q\}$.

Example

- $\mathrm{Aut}(T_3)$ is the symmetric group Σ_3.

  $\varphi : T_3 \to T_3$ is a quandle automorphism if and only if it is bijective.

  $\mathrm{Inn}(T_3)$ is the trivial group which is generated by the identity permutation.

- Both of $\mathrm{Aut}(Q(4,1))$ and $\mathrm{Inn}(Q(4,1))$ are the alternating group A_4.

  B. Ho and S. Nelson, Matrices and finite quandles, Homology Homotopy Appl. **7(1)** *(2005) 197-208.*

  $\mathrm{Inn}(Q(4,1))$ is the group generated by the permutations (123), (032), (013), (012), the columns of the operation table of $Q(4,1)$.

Definition

Let G be a group and A an abelian group.

- A group 2-cocycle of G by A is a function $\phi : G \times G \to A$ satisfying
  1. $\phi(1_G, g) = \phi(g, 1_G) = 0$ for $g \in G$,
  2. $\phi(g, h) + \phi(gh, k) = \phi(h, k) + \phi(g, hk)$ for $g, h, k \in G$.
- Two group 2-cocycles ϕ and ϕ' are said to be group cohomologous if there exists a function $f : G \to A$ such that $\phi'(g, h) = \phi(g, h) + f(g) + f(h) - f(gh)$ for $g, h \in G$.

Remark

Note that the 2nd group cohomology group $H^2_{\mathrm{gp}}(G; A)$ is the group of 2-cocycles of G by A up to the group cohomologous condition.

Definition

Let G and N be groups. A group extension of G by N is a short exact sequence

$$E : 1_N \to N \to \tilde{G} \to G \to 1_G.$$

An extension is said to be central if N lies in the center of $\tilde{G}$.

In this talk, we only consider a central extension of a group G by an abelian group A

$$0 \to A \to \tilde{G} \to G \to 1.$$

Remark

Let ϕ be a group 2-cocycle of G by A. Then $G \times A$ is a group under the operation $(g, a)(h, b) = (gh, a + b + \phi(g, h))$.
We denote the group by $G \times_\phi A$.
For the group monomorphism $\iota : A \to G \times_\phi A$ defined by $\iota(a) = (1, a)$ and the group epimorphism $\pi : G \times_\phi A \to G$ defined by $\pi(g, a) = g$,

$$E; 0 \to A \xrightarrow{\iota} G \times_\phi A \xrightarrow{\pi} G \to 1$$

is a central extension.

Proposition

There is a one-to-one correspondence between the set of all group extensions of G by A (up to equivalence) and $H^2_{\mathrm{gp}}(G; A)$.

S. MacLane, Homology, Springer-Verlag 1963.

Definition

Let $(Q, \triangleleft)$ be a quandle and A an abelian group.

- A quandle 2-cocycle of Q by A is a function $\psi : Q \times Q \to A$ satisfying
  1. $\psi(x, x) = 0$ for $x \in Q$,
  2. $\psi(x, y) + \psi(x \triangleleft y, z) = \psi(x, z) + \psi(x \triangleleft z, y \triangleleft z)$ for $x, y, z \in Q$.
- Two quandle 2-cocycles ψ and ψ' are said to be quandle cohomologous if there exists a function $\eta : Q \to A$ such that $\psi'(x, y) = \psi(x, y) + \eta(x) - \eta(x \triangleleft y)$ for $x, y \in Q$.

Remark

Note that the 2nd quandle cohomology group $H^2_{\mathrm{q}}(Q; A)$ is the group of 2-cocycles of Q by A up to the quandle cohomologous condition.

Definition

Let $(Q, \triangleleft)$ be a quandle and A an abelian group. Let $\psi : Q \times Q \to A$ be a quandle 2-cocycle. Then $Q \times A$ is a quandle under the operation defined by $(x, a) \triangleleft (y, b) = (x \triangleleft y, a + \psi(x, y))$. The quandle is called by the abelian extension of Q by A with ψ and denoted by $E(Q, A, \psi)$.

J. S. Carter, M. Elhamdadi, M. A. Nikiforou and M. Saito, Extensions of quandles and cocycle knot invariants J. Knot Theory Ramificarions, **12** *(2003) 725-738.*

Example

Note that the following $\psi : Q(4,1) \times Q(4,1) \to \mathbb{Z}_2$ is a quandle 2-cocycle of $Q(4,1)$ by $\mathbb{Z}_2$.

The abelian extension of $Q(4,1)$ by $\mathbb{Z}_2$ by ψ is isomorphic to $Q(8,1)$ and the epimorphism $p : Q(8,1) \to Q(4,1)$ is defined by $\{0,4\} \mapsto 0$, $\{1,5\} \mapsto 1$, $\{2,6\} \mapsto 2$ and $\{3,7\} \mapsto 3$.

ψ	0	1	2	3
0	0	1	1	0
1	1	0	1	0
2	1	1	0	0
3	0	0	0	0

$Q(8,1)$	0	1	2	3	4	5	6	7
0	0	7	5	2	0	7	5	2
1	6	1	7	0	6	1	7	0
2	7	4	2	1	7	4	2	1
3	1	2	0	3	1	2	0	3
4	4	3	1	6	4	3	1	6
5	2	5	3	4	2	5	3	4
6	3	0	6	5	3	0	6	5
7	5	6	4	7	5	6	4	7

Proposition

There is a one-to-one correspondence between the set of all abelian extensions of Q by A (up to equivalence) and $H_{\mathrm{q}}^2(Q; A)$.

J. S. Carter, S. Kamada and M. Saito, Diagrammatic computations for quandles and cocycle knot invariants, https://arxiv.org/pdf/math/0102092.pdf

Let G be a group and $\varphi \in \mathrm{Aut}(G)$.
Then G is a quandle under the operation $g \triangleleft_{\varphi} h = \varphi(gh^{-1})h$.
In particular, if φ_{ζ} denotes the conjugation by $\zeta \in G$, then

$$g \triangleleft_{\varphi_{\zeta}} h = \zeta^{-1}(gh^{-1})\zeta h.$$

D. Joyce, A classifying invariant of knots, the knot quandle, J. Pure Appl. Alg., **23** *(1983) 37-65.*

S. V. Matveev, Distributive groupoids in knot theory, (Russian) Mat. Sb. (N. S.) **119(161)** *(1982) no.1, 78-88, 160. English Translation : Math. USSR-Sb.* **47** *(1984), no. 1, 73-83.*

Lemma

Let G be a group and A an abelian group. Let ζ be fixed in G. If $\phi : G \times G \to A$ is a group 2-cocycle, then $\widetilde{\phi}_\zeta : (G, \lhd_\zeta) \times (G, \lhd_\zeta) \to A$ defined by

$$\begin{aligned}\widetilde{\phi}_\zeta(g,h) = & -\phi(h,h^{-1}) - \phi(\zeta,\zeta^{-1}) + \phi(\zeta,h) + \phi(h^{-1},\zeta h) \\ & + \phi(g,h^{-1}\zeta h) + \phi(\zeta^{-1}, gh^{-1}\zeta h)\end{aligned}$$

is a quandle 2-cocycle.

Proof.

For the group 2-cocycle ϕ, $G \times_\phi A$ is the extended group of G by A and s is corresponding section.
Since $G \times_\phi A$ is a quandle under $\lhd_{(\zeta,0)}$, $s(g) \lhd_{(\zeta,0)} s(h)$, $s(g \lhd_\zeta h) \in G \times_\phi A$.
Then $\pi([s(g) \lhd_{(\zeta,0)} s(h)]s(g \lhd_\zeta h)^{-1}) = 0$ and $[s(g) \lhd_{(\zeta,0)} s(h)]s(g \lhd_\zeta h)^{-1} \in A$.

From the identities $s(gh)^{-1} = s(h)^{-1}s(g)^{-1}\phi(g,h)$ and $s(h^{-1})^{-1} = \phi(h,h^{-1})^{-1}s(h)$,

$$
\begin{aligned}
& s(g) \lhd_{(\zeta,0)} s(h)[s(g \lhd_\zeta h)]^{-1} \\
&= (\zeta,0)^{-1}s(g)s(h)^{-1}(\zeta,0)s(h)[s(\zeta^{-1}gh^{-1}\zeta h)]^{-1} \\
&= (\zeta,0)^{-1}s(g)^{-1}s(h)^{-1}s(h^{-1})^{-1}s(g)s(\zeta^{-1})^{-1}\phi(\zeta,h)\phi(h^{-1},\zeta h)\phi(g,h^{-1}\zeta h) \\
&\quad \phi(\zeta^{-1},gh^{-1}\zeta h) \\
&= [\phi(h,h^{-1})]^{-1}[\phi(\zeta,\zeta^{-1})]^{-1}\phi(\zeta,h)\phi(h^{-1},\zeta h)\phi(g,h^{-1}\zeta h)\phi(\zeta^{-1},gh^{-1}\zeta h).
\end{aligned}
$$

By using the group 2-cocycle condition of ϕ repeatedly,
$\widetilde{\phi}_\zeta(g,g) = 0$ for all $g \in G$ and
$\widetilde{\phi}_\zeta(g,h) - \widetilde{\phi}_\zeta(g,k) + \widetilde{\phi}_\zeta(g \lhd_\zeta h, k) - \widetilde{\phi}_\zeta(g \lhd_\zeta k, h \lhd_\zeta k) = 0$ for all $g, h, k \in G$. $\square$

Theorem

The function $\Phi_\zeta : H^2_{\mathrm{gp}}(G; A) \to H^2_{\mathrm{q}}((G, \triangleleft_\zeta); A)$ define by $\Phi_\zeta(\phi) = \widetilde{\phi}_\zeta$ is a group homomorphism.

Proof.

Suppose that two group 2-cocycles ϕ and ϕ' are cohomologous.
Then there exists a function $f : G \to A$ such that
$\phi'(g, h) = \phi(g, h) + f(g) + f(h) - f(gh)$ for $g, h \in G$.
One can see that $\widetilde{\phi}_\zeta$ and $\widetilde{\phi'}_\zeta$ are cohomologous as quandle 2-cocycles by the same function f.
Hence Φ_ζ is well-defined.

For two group 2-cocyles ϕ and ϕ', it is not difficult to show that
$\Phi_\zeta(\phi + \phi')(g, h) = \Phi_\zeta(\phi)(g, h) + \Phi_\zeta(\phi')(g, h)$. □

Theorem

If $0 \to A \to G \times_\phi A \to G \to 1$ is a group extension via a group 2-cocycle ϕ, then the abelian extension $E((G, \lhd_\zeta), A, \Phi_\zeta(\phi))$ of $(G, \lhd_\zeta)$ by A corresponding to the quandle 2-cocycle $\Phi_\zeta(\phi)$ is the quandle $(G \times_\phi A, \lhd_{(\zeta,0)})$, the extended group $G \times_\phi A$ with the quandle operation $\lhd_{(\zeta,0)}$.

Proof.

$(G \times_\phi A, \lhd_{(\zeta,0)})$ and $E((G, \lhd_\zeta), A, \Phi_\zeta(\phi))$ have the same underlying set $G \times A$.

From the identity that $(g, a)^{-1} = (g^{-1}, -a - \phi(g, g^{-1}))$ and the group 2-cocycle condition, we can see that $(g, a) \lhd_{(\zeta,0)} (h, b) = (g \lhd_\zeta h, a + \Phi_\zeta(\phi)(g, h))$. $\square$

Example

Let G be a group and A an abelian group.

Consider the trivial group 2-cocycle $\phi_0 : G \times G \to A$, indeed it is defined by $\phi_0(g, h) = 0$ for all $g, h \in G$.

Then $\Phi_\zeta(\phi_0)$ is the trivial quandle 2-cocycle for any $\zeta \in G$ so that the abelian extension $E((G, \triangleleft_\zeta), A, \Phi_\zeta(\phi_0))$ is the trivial abelian extension of $(G, \triangleleft_\zeta)$ by A.

Note that $G \times_{\phi_0} A = G \times A$ is the cartesian product of two groups G and A, and $E((G, \triangleleft_\zeta), A, \Phi_\zeta(\phi_0))$ is the cartesian product of the quandle $(G, \triangleleft_\zeta)$ and the trivial quandle A which coincides $(G \times A, \triangleleft_{(\zeta,0)})$.

Example

It is known that $H^2_{\mathrm{gp}}(\mathbb{Z}_2;\mathbb{Z}_2) \cong \mathbb{Z}_2$ and $H^2_{\mathrm{q}}((\mathbb{Z}_2, \lhd_\zeta);\mathbb{Z}_2) \cong (\mathbb{Z}_2)^{16}$.
The only non-trivial central extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_2$ is

$$0 \to \mathbb{Z}_2 \xrightarrow{\iota} \mathbb{Z}_4 \xrightarrow{\pi} \mathbb{Z}_2 \to 0.$$

By choosing the section $s : \mathbb{Z}_2 \to \mathbb{Z}_4$ given by $s(0) = 0, s(1) = 1$, $\phi(g,h) = s(g) + s(h) - s(g+h)$ is a group 2-cocycle corresponding to the non-trivial central extension.

$$\Phi_0(\phi)(g,h) = -\phi(0,0) - \phi(h,-h) + \phi(0,h) + \phi(-h,h) + \phi(g,0) + \phi(0,g) = 0$$
$$\Phi_1(\phi)(g,h) = -\phi(1,1) - \phi(h,-h) + \phi(1,h) + \phi(-h,1+h) + \phi(g,1) + \phi(1,g+1) = 0$$

For the generator $[\phi]$ of $H^2_{\mathrm{gp}}(\mathbb{Z}_2;\mathbb{Z}_2)$, $\Phi_\zeta(\phi) = 0$ for any $\zeta \in \mathbb{Z}_2$.
Hence the group homomorphism Φ_ζ is neither injective nor surjective, in general.

Example

It is known that $H^2_{\mathrm{gp}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
One of non-trivial central extensions is that

$$0 \to \mathbb{Z}_2 \xrightarrow{\iota} Q_8 \xrightarrow{\pi} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0,$$

where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group, while ι is defined by $\iota(0) = 1, \iota(1) = -1$ and π is defined by $\pi(\pm 1) = (0, 0)$, $\pi(\pm i) = (0, 1)$, $\pi(\pm j) = (1, 0)$, $\pi(\pm k) = (1, 1)$.
By choosing the section $s : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to Q_8$ given by $s(0, 0) = 1, s(0, 1) = i, s(1, 0) = j, s(1, 1) = k$, we get the following group 2-cocycle ϕ which is corresponding to the non-trivial extension.

ϕ	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	0	0	0	0
(0,1)	0	1	0	1
(1,0)	0	1	1	0
(1,1)	0	0	1	1

For $(0,1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, we get the following table for the quandle 2-cocycle $\Phi_{(0,1)}(\phi)$.

$\Phi_{(0,1)}(\phi)$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	0	0	1	1
(0,1)	0	0	1	1
(1,0)	1	1	0	0
(1,1)	1	1	0	0

The operation of $E((\mathbb{Z}_2 \oplus \mathbb{Z}_2, \triangleleft_{(0,1)}), \mathbb{Z}_2, \Phi_{(0,1)}(\phi))$ is obtained as following table, where $(g, 0)$ corresponds to $s(g)$ and $(g, 1)$ corresponds to $-s(g)$ for each $g \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	1	-1	-1	1	1	-1	-1
i	i	i	$-i$	$-i$	i	i	$-i$	$-i$
j	$-j$	$-j$	j	j	$-j$	$-j$	j	j
k	$-k$	$-k$	k	k	$-k$	$-k$	k	k
-1	-1	-1	1	1	-1	-1	1	1
$-i$	$-i$	$-i$	i	i	$-i$	$-i$	i	i
$-j$	j	j	$-j$	$-j$	j	j	$-j$	$-j$
$-k$	k	k	$-k$	$-k$	k	k	$-k$	$-k$

We claim that $E((\mathbb{Z}_2 \oplus \mathbb{Z}_2, \triangleleft_{(0,1)}), \mathbb{Z}_2, \Phi_{(0,1)}(\phi))$ is not isomorphic to the trivial abelian extension of $(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \triangleleft_{(0,1)})$ by $\mathbb{Z}_2$.

$(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \triangleleft_{(0,1)})$ is isomorphic to the trivial quandle T_4, because $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is abelian.

Since any trivial abelian extension of a trivial quandle is also trivial, the trivial extension of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ by $\mathbb{Z}_2$ has eight orbits.

But, $E((\mathbb{Z}_2 \oplus \mathbb{Z}_2, \triangleleft_{(0,1)}), \mathbb{Z}_2, \Phi_{(0,1)}(\phi))$ has only four orbits.

By the proposition before, $\Phi_{(0,1)}(\phi)$ is not cohomologous to the trivial quandle 2-cocycle.

For a generator $[\phi]$ of $H^2_{\mathrm{gp}}(\mathbb{Z}_2 \oplus \mathbb{Z}_2; \mathbb{Z}_2)$, $\Phi_{(0,1)}(\phi) \neq 0$, so that $\Phi_{(0,1)}$ is a non-trivial group homomorphism.

Example

It is known that $H^2_{\mathrm{gp}}(A_4; \mathbb{Z}_2) \cong \mathbb{Z}_2$.
The only non-trivial extensions is that

$$0 \to \mathbb{Z}_2 \xrightarrow{\iota} \mathrm{SL}(2, \mathbb{Z}_3) \xrightarrow{\pi} A_4 \to 1,$$

because A_4 is isomorphic to $\mathrm{PSL}(2, \mathbb{Z}_3)$.
Note that $\mathrm{SL}(2, \mathbb{Z}_3)$ has the permutation presentation.
We choose a section $s : A_4 \to \mathrm{SL}(2, \mathbb{Z}_3)$ by

$$\begin{array}{ccc} s(0) = (0), & s(123) = (163)(275), & s(132) = (136)(257), \\ s(032) = (076)(243), & s(023) = (067)(234), & s(013) = (053)(174), \\ s(031) = (035)(147), & s(021) = (021)(465), & s(012) = (012)(456), \\ s(02)(13) = (0246)(1357), & s(01)(23) = (0541)(2367), & s(03)(12) = (0347)(1652). \end{array}$$

.

Y. Bae, J. S. Carter and B. Kim, Amusing permutation representations of group extensions, https://arxiv.org/pdf/1812.08475.pdf

From the section s, we have the quandle 2-cocycle $\phi : A_4 \times A_4 \to \mathbb{Z}_2$ defined by $\phi(g,h) = s(g)s(h)s(gh)^{-1}$ and it is corresponding to the non-trivial extension, while $\mathrm{SL}(2,\mathbb{Z}_3) \cong A_4 \times_\phi \mathbb{Z}_2$.

For $(123) \in A_4$, We can construct the quandle 2-cocycle $\Phi_{(123)}(\phi)$ and the abelian extension $E((A_4, \triangleleft_{(123)}), \mathbb{Z}_2, \Phi_{(123)})$ which coincides with $(\mathrm{SL}(2,\mathbb{Z}_3), \triangleleft_{(163)(275)})$.

By observing orbits of $(\mathrm{SL}(2,\mathbb{Z}_3), \triangleleft_{(163)(275)})$, we had the conclusion that $E((A_4, \triangleleft_{(123)}), \mathbb{Z}_2, \Phi_{(123)})$ was not trivial.

That is, $\Phi_{(123)}(\phi) \neq 0$ and $\Phi_{(123)}$ is a non-trivial group homomorphism.

Introduction of other work

- For a quandle Q, we have the inner automorphism group $\mathrm{Inn}(Q)$.
- Fix $z \in Q$ and $\zeta = (- \triangleleft z)$.
- By Joyce and Matveev, $(\mathrm{Inn}(Q), \triangleleft_\zeta)$ is a quandle and there is a quandle epimorphism $(\mathrm{Inn}(Q), \triangleleft_\zeta) \twoheadrightarrow Q$.
- We have a group homomorphism $H^2_{\mathrm{q}}(Q; A) \to H^2_{\mathrm{q}}((\mathrm{Inn}(Q), \triangleleft_\zeta); A)$ and the following diagram commutes.

$$\begin{tikzcd}
E(\mathrm{Inn}(Q), A, \Psi(\psi)) \arrow[r, "\hat{p}"] \arrow[d, "E_z \times Id_A"'] & \mathrm{Inn}(Q) \arrow[d, "E_z"'] \\
E(Q, A, \psi) \arrow[r, "p"] & Q
\end{tikzcd}$$

$$\begin{tikzcd} & H^2_{gp}(\mathrm{Inn}(Q);A) \arrow[d, "\Phi_\zeta"] \\ H^2_q(Q;A) \arrow[r, "\Psi_\zeta"] & H^2_q((\mathrm{Inn}(Q),\triangleleft_\zeta);A) \end{tikzcd}$$

Thanks for listening.