Projective characters for loops

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Projective representations for groups

A projective representation of a group G is a set of matrices $\{\kappa(g)\}_{g\in G}$ such that

$$\kappa(g)\kappa(h) = \alpha(g,h)\kappa(gh),$$

where $\alpha(g,h)$ is a scalar. Informally, the matrices are projectively equivalent. $\alpha(g,h)$ is called a **factor set.**
Example: $G = V_4 = \{e, a, b, ab\}$. This has a projective representation κ of degree 2:

$$\kappa(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \kappa(a) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \kappa(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \kappa(ab) = \begin{bmatrix} 0 & \\ -i & \end{bmatrix}$$

The representations κ_1 and κ_2 are **equivalent** if there exist elements $c(g_i)$ in $\mathbb{C}$ and an invertible matrix P in $M_m(K)$ such that

$$\kappa_1(g) = c(g)P^{-1}\kappa_2(g)P$$

for all $g \in G$. The associativity condition implies that each factor set α must satisfy

$$\alpha(g_1, g_2)\alpha(g_1g_2, g_3) = \alpha(g_1, g_2g_3)\alpha(g_2, g_3)$$

and if κ_1 and κ_2 are equivalent as above their factor sets α_1, α_2 are related by

$$\alpha_1(g_1, g_2) = c(g_1, g_2)\alpha_2(g_1, g_2)$$

where

$$c(g_1, g_2) = \frac{c(g_1)c(g_2)}{c(g_1g_2)}.$$

In the language of cohomology, α is a 2-cocycle and $c(g_1, g_2)$ is a coboundary.

Three approaches to the Schur multiplier
(1) The cocycles modulo the coboundaries
(2) A **Stem extension** of a group G is an exact sequence

$$\{0\} \to N \to H \to G \to \{e\}$$

where $N \subseteq G' \cap Z(G)$. The Schur multiplier is the (unique) N where $|H|$ is maximal.
In this case H is a **Darstellungsgruppe** for G (or a **Schur cover**).
(3) If

$$\{e\} \to R \to F \to G \to \{e\}$$

is a free presentation of the group G then

$$M \simeq \frac{R \cap F'}{[R, F]}.$$

General remarks about projective representations of groups

(1) A lot of the results are in the original papers of Schur.
(2) There are two volumes of Karpilovsky on projective representations
(these are intimidating!)
However, reviewers suggest that it is hard to sort out the proofs which are translations of those for linear representations and those which are difficult.
(3) They have appeared in the theory of finite frames.

Reasons to consider the extension to loops

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(1) Insight into the group theory
(2) For which varieties of loops do Schur covers exist?
(3) How do the three ways of looking at the Schur multiplier relate?

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What happens for Moufang loops?

(1) Factor sets:
The following relation is imposed on a factor set for a Moufang loop in the same way that the cocycle condition is obtained for groups:

$$f(z,y)f(x,zy)f(z,(x(zy)) = f(z,x)f(zx,z)f(((zx)z),y).$$

(2) If a stem extension is defined by the exact sequence of Moufang loops

$$\{0\} \to N \to H \to Q \to \{e\}$$

with $N \subseteq H' \cap Z(H)$ is there a maximum stem extension (a "Schur cover")?

(3) Suppose

$$\{e\} \to R \to F_M \to Q \to \{e\}$$

is now an exact sequence in the category of Moufang loops. Here F_M is a free Moufang loop and Q is a Moufang loop.
Is there a formula for " $\frac{R \cap F_M'}{[R,F]}$ "?

Example

Let G be the Klein group $V_4 = \{e, u, v, uv\}$. The dihedral group D_4 is a covering group for G, i.e. there is a projection $\mu : D_4 \to G$ with kernel M, the Schur multiplier. In this case if

$$D_8 =< a, b : a^4 = b^2 = e, bab = a^3 >$$

then $M = \{e, a^2\}$. The character table of D_4 is

Class	$\{e\}$	$\{a^2\}$	$\{a, a^3\}$	$\{b, a^2 b\}$	$\{ab, a^3 b\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

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The characters $\chi_1, ..., \chi_4$ correspond to ordinary representations of G and χ_5 corresponds to a projective representation κ of G.

Projective characters for groups

1) Can choose a unitary cocycle α ($\alpha(g,h)^m = 1$ for all g,h in G)
2) Define the character $\chi_\kappa(g)$ as $Tr(\kappa(g))$.
3) For unitary representations

$$\chi_\kappa(g^{-1}) = \alpha(g,g^{-1})\overline{\chi_\kappa(g)}.$$

4) Suppose κ_1 and κ_2 are projective reps with the same cocycle α. Then
(a) If $\chi_{\kappa_1} = \chi_{\kappa_2}$ then $\kappa_1 \cong \kappa_2$. Define $< \phi, \psi >= \frac{1}{|G|}\sum_{g \in G} \phi(g)\overline{\psi(g)}$.
(b) $< \chi_{\kappa_1}, \chi_{\kappa_2} >= Dim_{\mathbb{C}}(Hom_G(\kappa_1, \kappa_2))$
(c) If G is abelian then all finite dimensional irreducible projective representations of G with cocycle α have the same degree.

5) The α-**regular representation** R of G has a basis $\{e_g\}_{g\in G}$ such that

$$R(h)(e_g) = \alpha(h,g)e_{hg}.$$

6) R decomposes as $\bigoplus_{\kappa\in \mathrm{Rep}_G^\alpha} \kappa^{\oplus \deg(\kappa)}$

7) There is a space $\mathbb{H}_\alpha$ of α-class functions on G. A subset $\mathbb{L}_\alpha$ of the conjugacy classes of G is defined (the α-classes) and the number of inequivalent irreducible projective representations with cocycle α is $l_\alpha = |\mathbb{L}_\alpha|$.

8) Induced α-characters are defined. A Mackey criterion is available to decide whether an induced character is irreducible.

Some things to consider

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(1) Questions on general projective characters (for example in association schemes) could arise.
(2) In some senses projective representations are more "natural" than ordinary representations of groups
(3) The Schur multiplier is an example of a Baer invariant. These have been used to obtain cohomology theories. It seems intriguing to investigate these for varieties of Moufang loops.
(4) Stephen Gagola III has a theory of representations of (some) Moufang loops by Zorn matrices. Actual projective representations should exist for these loops.

Some examples in the loop case

Take the Chein loop $MG2$ corresponding to the group G. If D is a Schur cover of G then $MD2$ gives rise to a stem extension of G. This need not be maximal.

If G is the Klein 4-group then as is well-known $MG2 = C_2 \times C_2 \times C_2$ which has a Schur multiplier also isomorphic to $C_2 \times C_2 \times C_2$. Thus a Schur cover of $MG2$ has order 64. A computer search has found that there are many Moufang loops M_i of order 64 which may be regarded as loop covers in that there is an exact sequence

$$\{0\} \to M \to M_i \to MG2 \to \{e\}$$

with $M \subseteq G' \cap Z(G)$. It seems as though a general result is that for an arbitrary group G with Schur cover D the loop $MD2$ gives rise to a stem cover of $MG2$.

The Baer invariant

If $\mathcal{V}$ is a variety of groups defined by a set $U = \{u_i\}_{i=1}^{r}$ of words in variables $\{x_j\}_{j=1}^{t}$ and G is an arbitary group, the **verbal subgroup** $V(G)$ of G is the subgroup of G which is generated by the values of the words in U on sets of elements of G. The **marginal subgroup** $\mathcal{V}M(G)$ of G is the subgroup generated by the elements g such that for each word $u_i(x_1, x_2, ..., x_s)$

$$u_i(x_1 g, x_2, ..., x_s) = u_i(x_1, x_2 g, ..., x_s) = ... = u_i(x_1, x_2, ..., x_s g).$$

For a free presentation

$$\{e\} \to R \to F \to G \to \{e\}$$

define the subgroup $[RV^*F]$ as the subgroup generated by

$$u_i(f_1, f_2, ..., f_i r, f_{i+1}, ..., f_s)[u_i(f_1, f_2, ..., f_i, f_{i+1}, ..., f_s)]^{-1}$$

for all $r \in R$, $f_i \in F$, $u_i \in U$. Then the Baer invariant is

$$\frac{R \cap V(F)}{[RV^*F]}.$$

If $\mathcal{V}$ is the variety of abelian groups then the Baer invariant is precisely

$$\frac{R \cap F'}{[R, F]}.$$

If $\mathcal{V}$ is the variety of groups pof nilpotency class c then the Baer invrriant is

$$\frac{R \cap \gamma_{c+1}(F)}{[R,_c F]}$$

This has been called the c-nilpotent multiplier.

Schur-Baer varieties

$\mathcal{V}$ is called a **Schur-Baer variety** if for any group G for which the marginal factor group $G/V^*(G)$ is finite then $V(G)$ is finite and $|V(G)|$ divides the a power of $|G/V^*(G)|$.
Schur proved that $\mathcal{A}$ is a Schur-Baer variety.
The following are equivalent
(1) $\mathcal{V}$ is a **Schur-Baer variety**
(2) For every finite group G then the Baer invariant $\mathcal{V}M(G)$ hs order dividing a power of $|G|$.

Some questions
(1) What is the "loop multiplier" of a commutative Moufang loop?
(2) If the variety of commutative Moufang loops is substituted for $\mathcal{A}$, does this change the Baer invariant?
etc, etc,