Applications of non-associative Hopf algebras to Loop theory

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Linearizing

- O. Loos: *Symmetric spaces I: General theory* (1969).

  Idea of non-associative Hopf algebras.

- J.-P. Serre: *Lie algebras and Lie groups* (1964).

  He discuss local Lie groups by using distributions with support at e.

Distributions with support at a point

Let (Q, e) be a smooth pointed manifold and $(U, (x^1, \ldots, x^n))$ a coordinate neighborhood at e and the vector space

$$\mathcal{D}_e(Q) = \bigoplus_{e_1, \ldots, e_n \geq 0} \mathbb{R}\langle \partial_1^{e_1} \cdots \partial_n^{e_n}|_e \rangle$$

with $\partial_1^0 \cdots \partial_n^0|_e(f) := \delta_e := f(e)$.

Given $\varphi\colon Q_1 \to Q_2$, $e_2 = \varphi(e_1)$ a smooth map, consider the linear map

$$\begin{array}{rcl} \varphi' : \mathcal{D}_{e_1}(Q_1) & \to & \mathcal{D}_{e_2}(Q_2) \\ \mu & \mapsto & \varphi'(\mu)\colon g \mapsto \mu(g \circ \varphi) \end{array}$$

We obtain a functor from the category of smooth pointed manifolds to the category of vector spaces (**linearization**)

Manifolds Distributions

$$T_eQ = \mathbb{R}\langle \partial_1|_e, \dots, \partial_n|_e \rangle = \mathrm{Prim}(\mathcal{D}_e(Q)) = \{\mu | \Delta(\mu) = \mu \otimes \delta_e + \delta_e \otimes \mu\}.$$

Manifolds

Diagonal

$$\begin{array}{rcl} Q & \to & Q \times Q \\ x & \mapsto & (x, x) \end{array}$$

Distributions

Comultiplication

$$\begin{array}{rcl} \Delta\colon \mathcal{D}_e(Q) & \to & \mathcal{D}_{(e,e)}(Q \times Q) \cong \mathcal{D}_e(Q) \otimes \mathcal{D}_e(Q) \\ \mu & \mapsto & \sum \mu_{(1)} \otimes \mu_{(2)} \end{array}$$

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Counit

$$\begin{array}{rcl} \epsilon\colon \mathcal{D}_e(Q) & \to & \mathcal{D}_e(e) \cong \mathbb{R} \\ \mu & \mapsto & \epsilon(\mu) \end{array}$$

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Antipode

$$\begin{aligned} \mathcal{D}_e(Q) &\to \mathcal{D}_e(Q) \\ \mu &\mapsto S(\mu) \end{aligned}$$

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1. (Bialgebra) Unital algebra + Δ and ϵ homomorphisms of unital algebras

To prove it for $\mathcal{D}_e(Q)$

linearize. . .

$$\begin{array}{ccc} (x, y) & \to & xy \\ \downarrow & & \downarrow \\ (x, x, y, y) & \to & (xy, xy) \end{array}$$

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$$\begin{array}{ccc} (x,y) & \to & (x,y,y) \\ \downarrow & & \downarrow \\ x & \sim & (xy)/y \end{array}$$

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- *(Coasociativity)* $(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta$

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- *(Coasociativity)* $(\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta$
- *(Cocommutativity)* $\Delta = \Delta^{op}$

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If Q is a group

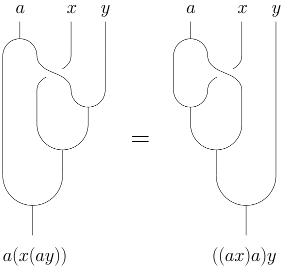
$$\begin{array}{ccc} (x, y, z) & \rightarrow & (x, yz) \\ \downarrow & & \downarrow \\ (xy, z) & \rightarrow & (xy)z = x(yz) \end{array}$$

linearizing...

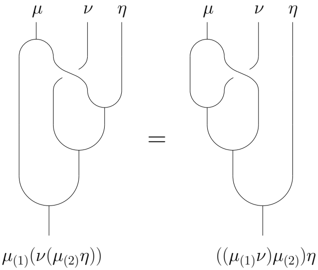
$$\begin{array}{ccc} \mu \otimes \nu \otimes \eta & \rightarrow & \mu \otimes \nu\eta \\ \downarrow & & \downarrow \\ \mu\nu \otimes \eta & \rightarrow & (\mu\nu)\eta = \mu(\nu\eta) \end{array}$$

$\mathcal{D}_e(Q)$ is associative

If Q is a Moufang loop



linearizing. . .



Hopf-Moufang

If Q is. . .

linearizing. . .

More interactions
between
Hopf like objects
and
non-associative algebra

Definition (J.D.H. Smith, 2016)

A quantum quasigroup (resp. quantum loop) in a symmetric monoidal category $(\mathbf{V}, \otimes, \mathbf{1})$ is a bimagma (resp. biunital bimagma) (A, ∇, Δ) in $\mathbf{V}$ for which the left composite morphism

$$A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \nabla} A \otimes A$$

and its dual composite

$$A \otimes A \xrightarrow{1_A \otimes \Delta} A \otimes A \otimes A \xrightarrow{\nabla \otimes 1_A} A \otimes A$$

are both invertible.

This definition is selfdual

Tangent space = Primitive elements

Tangent algebras of certain local smooth loops

Lie's theorems

The categories of local Lie groups and f.d. real Lie algebras are equivalent

Malcev, 1955; Kuzmin 1970

The categories of local smooth Moufang loops and f.d. real Malcev algebras are equivalent
It was made global by Kerdman (1979) and P. Nagy (1993)

Mikheev and Sabinin, 1982

The categories of local smooth Bol loops and f.d. real Bol algebras are equivalent

Moufang loop: $a(x(ay)) = ((ax)a)y$

Bol loop: $a(x(ay)) = (a(xa))y$

Bol algebra: a vector space with a skew-commutative product $[x, y]$ and a trilinear product $[x, y, z]$ that satisfy

$$[a, a, b] = 0$$
$$[a, b, c] + [b, c, a] + [c, a, b] = 0$$
$$[x, y, [a, b, c]] = [[x, y, a], b, c] + [a, [x, y, b], c] + [a, b, [x, y, c]]$$
$$[a, b, [c, d]] = [[a, b, c], d] + [c, [a, b, d]] + [c, d, [a, b]] + [[a, b], [c, d]]$$

Tangent algebras of general local smooth loops

- Kikkawa: *On local loops in affine manifolds* (1964).
- Sabinin: *The geometry of loops* (1972).

  Geodesic loops.

Tangent algebras of general local smooth loops

- Yamaguti:

  On locally reductive spaces and tangent algebras (1972).
  Geometry of homogeneous Lie loops (1975).

Tangent algebras of general local smooth loops

- Akivis: *The local algebras of a multidimensional three-web* (1976).
  Akivis algebras.

Tangent algebras of general local smooth loops

- Hofmann and Strambach: *Lie's fundamental theorems of local analytical loops* (1986).

  General approach based on Akivis algebras.

Tangent algebras of general local smooth loops

- Mikheev and Sabinin: *Infinitesimal theory of local analytic loops* (1986).
  Complete solution to the general case. Based on Sabinin algebras.

Tangent algebras of general local smooth loops

- Figula: *Geodesic loops* (2000).

  General approach to geodesic loops based on Λ-algebras.

Tangent algebras of general local smooth loops

- Weingart: *On the axioms for Sabinin algebras* (2016).

  General approach to geodesic loops based on a new definition of Sabinin algebra.

Tangent algebras of general local smooth loops

- Too many relevant papers and monographs from the study of specific varieties of loops or from other areas (webs, quasigroups, topological loops,...) have contributed to the understanding of the problem.
- At some point it was clear that a geodesic loop was nothing but an affine connection with zero curvature. Thus, one can define operations on the tangent space to either
  - model the torsion ($\Rightarrow$ many identities) or
  - the fundamental vector fields ($\Rightarrow$ very simple definition).

Mikheev and Sabinin, 1987

Local loops are classified by Sabinin algebras

Sabinin algebra: a vector space with two families of multilinear operations $\langle x_1, \ldots, x_n; x, y\rangle$ $(n \geq 0)$ and $\Phi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ $n \geq 1, m \geq 2$ that satisfy

$$\langle x_1, x_2, \ldots, x_m; y, z\rangle = -\langle x_1, x_2, \ldots, x_m; z, y\rangle,$$

$$\langle x_1, x_2, \ldots, x_r, a, b, x_{r+1}, \ldots, x_m; y, z\rangle - \langle x_1, x_2, \ldots, x_r, b, a, x_{r+1}, \ldots, x_m; y, z\rangle$$
$$+ \sum_{k=0}^{r} \sum_{\alpha} \langle x_{\alpha_1}, \ldots, x_{\alpha_k}, \langle x_{\alpha_{k+1}}, \ldots, x_{\alpha_r}; a, b\rangle, \ldots, x_m; y, z\rangle = 0,$$

$$\sigma_{x,y,z}\left(\langle x_1, \ldots, x_r, x; y, z\rangle + \sum_{k=0}^{r} \sum_{\alpha} \langle x_{\alpha_1}, \ldots, x_{\alpha_k}; \langle x_{\alpha_{k+1}}, \ldots, x_{\alpha_r}; y, z\rangle, x\rangle\right) = 0,$$

$\sigma_{x,y,z}$ denotes the cyclic sum on x, y, z

$\Phi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ is symmetric on $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$

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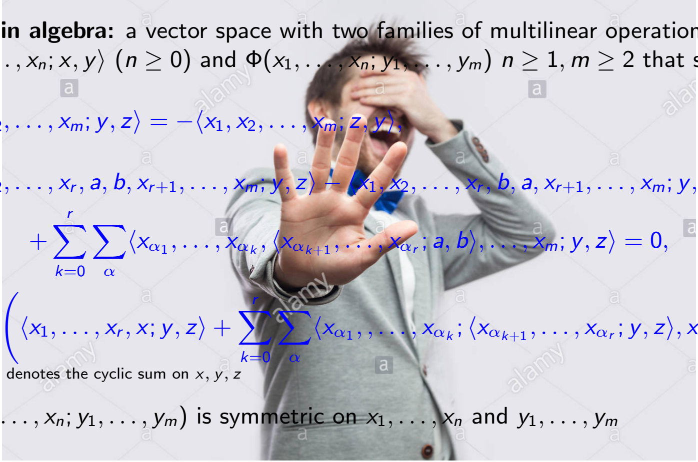

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Why so many operations?

Operations for primitive elements

Let $k\{V\}$ be the (unital) free non-associative algebra on a basis of V. The maps $a \mapsto a \otimes 1 + 1 \otimes a$ and $a \mapsto 0$ $(a \in V)$ induce homomorphisms

$$\begin{aligned} \Delta\colon k\{V\} &\to k\{V\} \otimes k\{V\} \\ u &\mapsto \sum u_{(1)} \otimes u_{(2)} \end{aligned}$$

and $\epsilon\colon k\{V\} \to k$ so that we get a non-associative Hopf algebra.

Is V the space of all primitive elements? NO.

$$T_e Q = \mathbb{R}\langle \partial_1|_e, \dots, \partial_n|_e \rangle = \mathrm{Prim}(\mathcal{D}_e(Q)) = \{\mu | \Delta(\mu) = \mu \otimes \delta_e + \delta_e \otimes \mu\}.$$

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Is V the space of all primitive elements? NO.

For any $a, a', b, c, ... \in V$

- Commutators: $[a, b]$
- Associators: (a, b, c)
- Other: $(aa', b, c) - a(a', b, c) - a'(a, b, c)$
- ...?

are primitive elements.

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Is V the space of all primitive elements? NO.

Problem: define operations so that V generates all primitive elements.

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Shestakov and Umirbaev, 2001

The space of primitive elements of a bialgebra is a Sabinin algebra

Shestakov-Umirbaev functor: consider the free unital algebra on $\{x_1, x_2, \ldots, y_1, y_2, \ldots, x\}$. From the associator (u, v, x) of $u = ((x_1 x_2)\cdots)x_n$, $v = ((y_1 y_2)\cdots)y_m$ and x define primitive elements

$$p(x_1, \ldots, x_n; y_1, \ldots, y_m; x) = p(u; v; x)$$

recursively by

$$p(u; v; x) := \sum (u_{(1)} v_{(1)}) \backslash (u_{(2)}, v_{(2)}, x)$$

$\sum u_{(1)} \otimes u_{(2)} = ((x_1 \otimes 1 + 1 \otimes x_1)(x_2 \otimes 1 + 1 \otimes x_2)) \cdots (x_n \otimes 1 + 1 \otimes x_n)$.

For any algebra A define

$$\begin{aligned}
\langle x, y\rangle &= -[x, y] \\
\langle x_1, \ldots, x_n; x, y\rangle &= p(x_1, \ldots, x_n; y; x) - p(x_1, \ldots, x_n; x; y) \quad (n \geq 1) \\
\Phi(x_1, \ldots, x_n; y_1, \ldots, y_m) &= \frac{1}{n!m!} \sum_{\sigma \in S_n, \tau \in S_m} p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; y_{\sigma(1)}, \ldots ; y_{\sigma(m)})
\end{aligned}$$

$A^{\text{УШ}} = (A, \langle ; , \rangle, \Phi)$ is a Sabinin algebra

$\mathrm{Prim}(k\{V\}) =$ Sabinin subalgebra of $k\{V\}^{\text{УШ}}$ generated by V

Universal enveloping algebras

- Associative algebra $A \Rightarrow$ Lie algebra $A^{(-)} = (A, [\ ,\ ])$
- Nonassociative algebra $A \Rightarrow$ Sabinin algebra $A^{\text{УШ}} = (A, \langle\ ;\ ,\ \rangle, \Phi)$

Conversely

- Lie algebra $\mathfrak{g} \Rightarrow \exists$ Hopf algebra $U(\mathfrak{g})$, $\mathfrak{g} \leq U(\mathfrak{g})^{(-)}$, $\mathfrak{g} = \mathrm{Prim}(U(\mathfrak{g}))$.
- Sabinin algebra $\mathfrak{s} \Rightarrow \exists$ non-associative Hopf algebra $U(\mathfrak{s})$, $\mathfrak{s} \leq U(\mathfrak{s})^{\text{УШ}}$, $\mathfrak{s} = \mathrm{Prim}(U(\mathfrak{s}))$.

Shestakov and P.I., 2004

Malcev algebras appear as primitive elements of Hopf-Moufang algebras

P.I., 2005

Bol algebras appear as primitive elements of Hopf-Bol algebras

P.I., 2007

Sabinin algebras appear as primitive elements of non-associative Hopf algebras

Some applications

Non-associative Hopf algebras allow computations with primitive elements, i.e. with tangent vectors

Commutative automorphic loops

Commutative A-loops

A loop is **commutative automorphic** if it satisfies the identities:

1. (*commutative*) $xy = yx$ and
2. (*left automorphic*)

$$(xy)\backslash(x(y(wz))) = ((xy)\backslash(x(yw)))\,((xy)\backslash(x(yz)))\,.$$

A **commutative automorphic Hopf algebra** is a non-associative Hopf algebra that sastisfies the identities:

1. (*commutative*) $xy = yx$ and
2. (*left automorphic*)

$$\begin{aligned}(x_{(1)}y_{(1)})\backslash(x_{(2)}(y_{(2)}(wz))) \\ = \quad ((x_{(1)}y_{(1)})\backslash(x_{(2)}(y_{(2)}w)))\,((x_{(3)}y_{(3)})\backslash(x_{(4)}(y_{(4)}z)))\,.\end{aligned}$$

What about T_eQ, i.e. $\mathrm{Prim}(\mathcal{D}_e(Q))$?

Studying $T_e Q$: case of Bruck loops.

$$((xy_{(1)})z)y_{(2)} = x((y_{(1)}z)y_{(2)}), \quad S(xy) = S(x)S(y), \quad S(x) := 1/x$$

$$a_{(1)} \otimes a_{(2)} = a \otimes 1 + 1 \otimes a, \quad \epsilon(a) = 0$$

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$$S(x_{(1)})x_{(2)} = \epsilon(x)1 \Rightarrow \left\{ \begin{array}{l} S(1) = 1 \\ S(a) = -a \\ S(ab) - ab - ba + ab = 0 \end{array} \right\} \Rightarrow S(ab) = ba \overset{S(ab)=ab}{\Rightarrow} ab = ba$$

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$$(xa)z + (xz)a = x(az) + x(za) \Rightarrow \left\{ \begin{array}{l} (x, a, z) = -(x, z, a) \\ \\ R_a(xz) = -R_a(x)z + x\,\overbrace{(L_a + R_a)}^{T_a}(z) \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} [R_a, R_b](xz) = [R_a, R_b](x)z + x[T_a, T_b](z) \\ \\ (\text{with } x = 1)\ [R_a, R_b] = [T_a, T_b] \end{array} \right. \Rightarrow [[R_a, R_b], R_z] = R_{[R_a, R_b](z)}$$

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Thus, $[R_a, [R_b, R_c]] = R_{[a,b,c]}$ with

$$[a, b, c] = -[R_b, R_c](a) = -(ac)b + (ab)c = (a, b, c) - (a, c, b) = 2(a, b, c)$$

$$\boxed{(T_eQ, [\ ,\ ,\ ]) \text{ is a Lie triple system}}$$

$$a_{(1)} \otimes a_{(2)} = a \otimes 1 + 1 \otimes a, \quad \epsilon(a) = 0$$

Commutative A-loops

Jedlička, Kinyon, Vojtěchovský (2011): if (Q, xy) is a uniquely 2-divisible commutative automorphic loop then

$$x \cdot y = P_{\sqrt{x}}(y)$$

with $P_x := L_{x^{-1}}^{-1} L_x = L_x L_{x^{-1}}^{-1}$ is a (left) Bruck loop.

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The tangent space of a formal commutative A-loop with the triple product

$$[a, b, c] = a(bc) - b(ac)$$

is a **commutative automorphic Lie triple system**

1. $[a, b, c] = -[b, a, c]$,
2. $[a, b, c] + [b, c, a] + [c, a, b] = 0$ and
3. $[[a, b, c], a', b'] = [[a, a', b'], b, c] + [a, [b, a', b'], c] + [a, b, [c, a', b']]$

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Grishkov, P.I. (2018): the category of formal commutative automorphic loops is equivalent to the category of commutative automorphic Lie triple systems.

Commutative A-loops

A **Baker-Campbell-Hausdorff formula** for formal commutative automorphic loops:

$$\mathrm{BCH}(a,b) = a + b + \sum_{i,j\geq 1} \beta_{i,j}[a, b, \overset{i-1}{\ldots} a, \overset{j-1}{\ldots} b]$$

where $\beta_{i,j}$ $(i,j \geq 1)$ is the coefficient of $s^i t^j$ in the Taylor expansion of

$$\frac{\left(e^{2s} - e^{2t}\right)(s+t)}{2\left(e^{2(s+t)} - 1\right)}$$

at $(0,0)$,

$$[a_1, a_2, \ldots, a_n] := \begin{cases} 0 & \text{if } n \text{ is even} \\ a_1 & \text{if } n = 1 \\ [[[a_1, a_2, a_3], \cdots], a_{n-1}, a_n] & \text{if } n > 1 \text{ is odd} \end{cases}$$

and

$$\overset{i}{\ldots} c := c, c, \ldots, c \quad \text{where } c \text{ appears } i \text{ times.}$$

A non-associative Baker-Campbell-Hausdorff formula

- P. T. Nagy and K. Strambach: *Loops, cores and symemtric spaces* (1998).
- P. O. Miheev and L. V. Sabinin: *The theory of smooth Bol loops* (1985).
- G. P. Nagy: *The Campbell-Hausdorff series of local analytic Bruck loops (2002). Problem raised, for local Bol loops, by Akivis and Goldberg in Loops '99.*
- A. Figula: *Geodesic loops* (2000).
- G. Weingart: *On the axioms for Sabinin algebras* (2016).

Baker-Campbel-Hausdorff

In the (complection of the) free associative algebra on x, y define

$$e(x) := \sum_{n\geq 0} \frac{1}{n!} x^n, \quad \log_{e(x)}(1+x) := \sum_{n\geq 1} (-1)^{n-1} \frac{x^n}{n}$$

We have $e(\log_{e(x)}(1+x)) = 1 + x$, $\log_{e(x)}(e(x)) = x$ and

$$\mathrm{BCH}(x, y) := \log_{e(x)}(e(x)e(y)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{r_i+s_i\geq 1} \frac{1}{r_1!s_1!\cdots r_n!s_n!} x^{r_1} y^{s_1} \cdots x^{r_n} y^{s_n}$$

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Dynkin-Specht-Wever lemma. Let $\gamma_d(u) := S(u_{(1)})d(u_{(2)})$ with $d(u) := |u|u$ for homogeneous u. Then $\gamma_d(a) = S(a)d(1) + S(1)d(a) = |a|a$ for homogeneous primitive a and

$$\begin{aligned}\gamma_d(ua) &= S(u_{(1)}a)d(u_{(2)}) + S(u_{(1)})d(u_{(2)}a)\\ &= -aS(u_{(1)})d(u_{(2)}) + S(u_{(2)})d(u_{(2)})a + \epsilon(u)a = [\gamma_d(u), a] + \epsilon(u)a.\end{aligned}$$

Baker-Campbel-Hausdorff

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$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{r_i+s_i\geq 1}\frac{(\sum_{j=1}^n(r_j+s_j))^{-1}}{r_1!s_1!\cdots r_n!s_n!}\gamma_d(x^{r_1}y^{s_1}\cdots x^{r_n}y^{s_n})$$
$$= x+y+\frac{1}{2}[x,y]+\frac{1}{12}[x,[x,y]]-\frac{1}{12}[y,[x,y]]+\frac{1}{24}[[x,[x,y]],y]+\dots$$

Dynkin-Specht-Wever lemma. Let $\gamma_d(u) := S(u_{(1)})d(u_{(2)})$ with $d(u) := |u|u$ for homogeneous u. Then $\gamma_d(a) = S(a)d(1)+S(1)d(a) = |a|a$ for homogeneous primitive a and

$$\gamma_d(ua) = S(u_{(1)}a)d(u_{(2)}) + S(u_{(1)})d(u_{(2)}a)$$
$$= -aS(u_{(1)})d(u_{(2)}) + S(u_{(2)})d(u_{(2)})a + \epsilon(u)a = [\gamma_d(u), a] + \epsilon(u)a.$$

Baker-Campbel-Hausdorff

When studying the differential equation

$$X(t)' = X(t)A(t)$$

in a Lie group, convert it in a differential equation in the Lie algebra, solve it and go back to the group. Formally, if $X(t) = e(\Omega(t))$ then

$$\Omega'(t) = \sum_{n \geq 0} \frac{B_n}{n!} \operatorname{ad}^n_{\Omega(t)}(A(t)) \qquad \text{(recurrence for } \Omega(t)\text{)}$$

In case $X(t) := e(x)e(ty)$, i.e. $\Omega(t) = \log_{e(x)}(e(x)e(ty))$

$$X(t)' = \exp(x)\big(\exp(ty)y\big) = \big(\exp(x)\exp(ty)\big)y = X(t)y.$$

and we can recursively compute $\Omega(1) = \mathrm{BCH}(x, y)$.

Baker-Campbel-Hausdorff

When studying the differential equation

$$X(t)' = X(t)A(t)$$

in a Lie group, convert it in a differential equation in the Lie algebra, solve it and go back to the group. Formally, if $X(t) = e(\Omega(t))$ then

In the **non-associative setting**, the problem is the same, the techniques are similar but the solution is much more messy

$$\Omega'(t) = \left(\tau^{e(\Omega(t))}\right)^{-1}(A(t)) \qquad (\text{recurrence for } \Omega(t))$$

where

$$\tau^{e(x)}(y) = e(x) \Big\backslash \left.\frac{d}{ds}\right|_{s=0} e(x+sy)$$

Non-associative Baker-Campbell-Hausdorff

In the (complection of the) free non-associative algebra on x, y,

$$e(x) := \exp_l(x) := \sum_{n\geq 0} \frac{1}{n!} \underbrace{(((xx)\cdots)x)x}_{n}, \quad \log_{e(x)}(1+x) := \log_l(1+x) := \sum_{\tau} \frac{B_\tau}{\tau!}\tau$$

Non-associative Baker-Campbell-Hausdorff

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For $X(t)' = X(t)A(t)$ with $X(t) = e(\Omega(t))$,

$$\Omega'(t) = \left(\tau^{e(\Omega(t))}\right)^{-1}(A(t)) \quad \text{with} \quad \tau^{e(x)}(y) = e(x) \Big\backslash \frac{d}{ds}\Big|_{s=0} e(x+sy)$$

Fortunately, $\tau^{e(x)}(y) = \gamma_{y\partial_x}(e(x))$ and we can use a non-associative DSW lemma to recursively compute its component τ_n of degree n in x by

$$\sum_{i=1}^{n} \frac{1}{n+1}\frac{1}{(n-i)!}\langle \underbrace{x,\ldots,x}_{n-i}; x, \tau_{i-1}\rangle$$

The particular case $X(t) := e(x)e(ty)$ corresponds to the BCH formula.

Non-associative Baker-Campbell-Hausdorff

In the (complection of the) free non-associative algebra on x, y,

$$e(x) := \exp_l(x) := \sum_{n\geq 0} \frac{1}{n!} \underbrace{(((xx)\cdots)x)x}_{n}, \quad \log_{e(x)}(1+x) := \log_l(1+x) := \sum_{\tau} \frac{B_\tau}{\tau!}\tau$$

$$\begin{aligned}
\mathrm{BCH}_l(x,y) &= \log_l(\exp_l(x)\exp_l(y)) = \\
&x + y + \frac{1}{2}[x,y] \\
&+ \frac{1}{12}[x,[x,y]] - \frac{1}{3}\langle x; x, y\rangle - \frac{1}{12}[y,[x,y]] - \frac{1}{6}\langle y; x, y\rangle - \frac{1}{2}\Phi(x; y, y) \\
&- \frac{1}{24}\langle x; x, [x,y]\rangle - \frac{1}{12}[x, \langle x; x, y\rangle] - \frac{1}{8}\langle x, x; x, y\rangle \\
&+ \frac{1}{24}[[x,[x,y]],y] - \frac{1}{24}[x,\langle y; x, y\rangle] - \frac{1}{4}\Phi(x,x;y,y) - \frac{1}{4}[x, \Phi(x;y,y)] \\
&- \frac{1}{24}[\langle x; x, y\rangle, y] - \frac{1}{24}\langle x; [x,y], y\rangle - \frac{1}{6}\langle x, y; x, y\rangle + \frac{1}{24}\langle y, x; x, y\rangle \\
&+ \frac{1}{12}[\Phi(x;y,y),y] + \frac{1}{24}\langle y; y, [x,y]\rangle - \frac{1}{24}\langle y, y; x, y\rangle - \frac{1}{6}\Phi(x;y,y,y) + \ldots
\end{aligned}$$

Free loops as non-associative power series

G. Higman, *The lower central series of a free loop* (1963)

Magnus homomorphism

Let $\mathcal{F} = \mathcal{F}(\boldsymbol{x})$ be the free loop on the set $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ and $\overline{\mathbb{Q}\{\boldsymbol{X}\}}$ the completion of the free unital non-associative algebra on the set $\boldsymbol{X}$ with coefficients in $\mathbb{Q}$.

Is the homomorphism $\begin{aligned} \mathcal{M}\colon \mathcal{F}(\boldsymbol{x}) &\to \overline{\mathbb{Q}\{\boldsymbol{X}\}}^{\times} \\ x_i &\mapsto 1 + X_i \end{aligned}$ injective?

Magnus homomorphism

Given $\mathcal{F}$ a free loop, N a normal subloop, Higman studied the relationship between $\mathcal{F}/N$ and $\mathcal{F}/[N,\mathcal{F}]$ by means of central extensions.

Given $\alpha\colon \mathcal{F} \to L := \mathcal{F}/N$, a central factorization of α is $\alpha = \gamma\beta$

$$\mathcal{F} \xrightarrow{\beta} M \xrightarrow{\gamma} L \quad \text{with} \quad \ker\gamma \subseteq Z(M)$$

Construction: let $(A,+)$ be a free abelian group with generators $f(l_1,l_2)$, $l_1,l_2 \in L$ different from e, and $g(x)$ for $x \in \boldsymbol{x}$. Then $(L,A) := L \times A$ is a loop with

$$(l_1,a_1)(l_2,a_2) = (l_1 l_2, a_1 + a_2 + f(l_1,l_2))$$

$(f(e,l_2) = 0 = f(l_1,e_2))$. The homomorphism

$$\begin{aligned} \delta\colon \mathcal{F} &\to (L,A) \\ x \in \boldsymbol{x} &\mapsto (\alpha(x), g(x)) \end{aligned}$$

defines a central factorization $\mathcal{F} \xrightarrow{\delta} (L,A) \to L$ of α.

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Higman (1963): The intersection of the terms of the lower central series of a free loop is the identity.

Magnus homomorphism

Based on the central extension

$$(l_1, a_1)(l_2, a_2) = (l_1 l_2, a_1 + a_2 + f(l_1, l_2)),$$

an adequate (completed) non-associative Hopf algebra $\overline{\mathbb{Q}\{\boldsymbol{X}\} \otimes \mathbb{Q}[T]}$ with product

$$(x \otimes \alpha)(y \otimes \beta) = \sum x_{(1)} y_{(1)} \otimes t^*(x_{(2)} \otimes y_{(2)})\alpha\beta.$$

where t^* is a certain coalgebra morphism, $m(\boldsymbol{X})$ is the set of all non-associative monomials on $\boldsymbol{X}$ and T is the set of symbols

$$\{t_1, \dots, t_n\} \sqcup \{t(m_1, m_2) \mid m_1, m_2 \in m(\boldsymbol{X})\},$$

and some results in Higman's paper we have

Mostovoy, P.I., Shestakov (2019):

$$\begin{aligned} \mathcal{M}\colon \mathcal{F}(\boldsymbol{x}) &\to \overline{\mathbb{Q}\{\boldsymbol{X}\}}^{\times} \qquad \text{is injective} \\ x_i &\mapsto 1 + X_i \end{aligned}$$

Thank you