# Highlights from the research of Jonathan D H Smith

Petr Vojtěchovský

University of Denver

Loops '19
Budapest University of Technology and Economics, Hungary
July 7–13, 2019

# Students

## Jonathan Dallas Hayden Smith

MathSciNet

Ph.D. University of Cambridge 1975

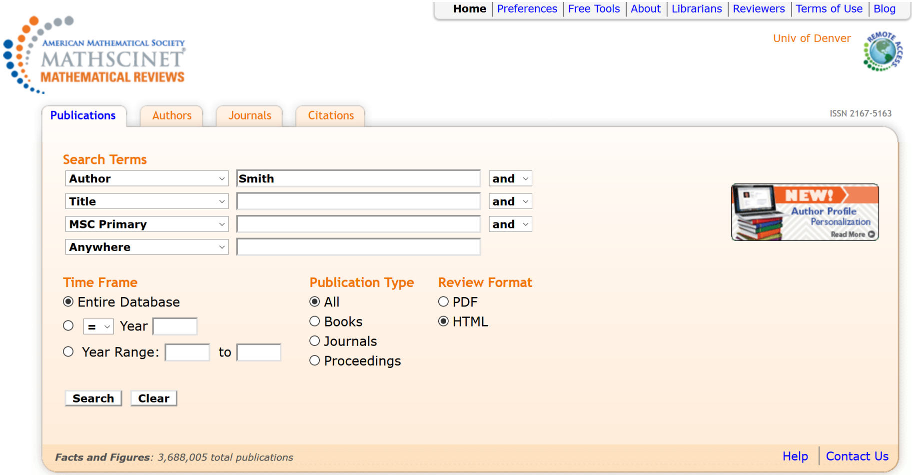
Dissertation: *Centrality*

Advisor: John Horton Conway

Students:
Click here to see the students listed in chronological order.

| Name | School | Year | Descendants |
|---|---|---|---|
| Aydinyan, Ruben | Iowa State University | 2005 | |
| Choi, Dug Hwan | Iowa State University | 1998 | |
| Choi, Ji Young | Iowa State University | 2002 | |
| Chung, Key One | Iowa State University | 2007 | |
| Dagli, Mehmet | Iowa State University | 2008 | |
| Fiedler, James | Iowa State University | 2007 | |
| Fuad, Tengku | Iowa State University | 1993 | |
| Hobart, Michael | Iowa State University | 1993 | |
| Hsu, Feng-Luan | Iowa State University | 1996 | |
| Hummer, Frank | Iowa State University | 1992 | |
| Kivunge, Benard | Iowa State University | 2004 | 3 |
| Mutungi, Patrick | Iowa State University | 2004 | |
| Phillips, J. D. | Iowa State University | 1992 | |
| Rice, Theodore | Iowa State University | 2007 | |
| Shen, Xiaorong | Iowa State University | 1991 | |
| Stines, Elijah | Iowa State University | 2012 | |
| Thur, Lois | Iowa State University | 1993 | |
| Vojtěchovský, Petr | Iowa State University | 2001 | 6 |
| Wang, Stefanie | Iowa State University | 2017 | |
| Wells, Andrew | Iowa State University | 2010 | |
| Wojdylo, Jerzy | Iowa State University | 1998 | |

# Publications



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**MR3975621** **Prelim** Bunimovich, Leonid; Smith, Dallas; Webb, Benjamin; Finding Hidden Structures, Hierarchies, and Cores in Networks via Isospectral Reduction. *Appl. Math. Nonlinear Sci.* 4 (2019), no. 1, 225–248. 94
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Note: Every 1 in 500 math papers is written by a Smith!

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| Website: | http://math.iastate.edu/jdhsmith/math/math_ |
| MR Author ID: | **163995** |
| Earliest Indexed Publication: | 1976 |
| Total Publications: | **179** |
| Total Related Publications: | **4** |
| Total Citations: | **875** |

⊞ Published as: Smith, J. (1)

# Books

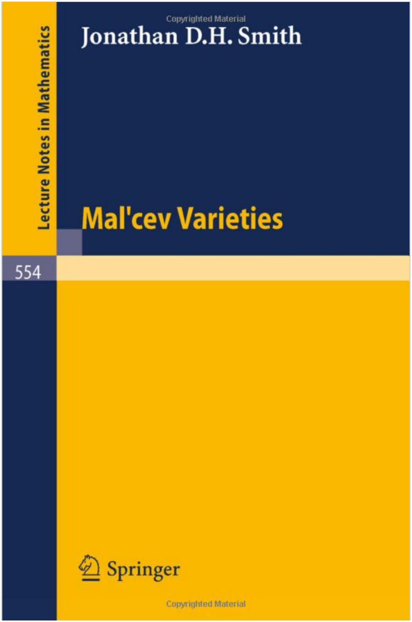
## Author Citations for Jonathan D. H. Smith

**Jonathan D. H. Smith is cited 875 times by 359 authors**

in the MR Citation Database

Most Cited Publications

| Citations | Publication | |
|---|---|---|
| 133 | **MR0432511 (55 #5499)** Smith, Jonathan D. H. Mal'cev varieties. Lecture Notes in Mathematics, Vol. 554. *Springer-Verlag, Berlin-New York,* 1976. viii+158 pp. (Reviewer: V. A. Artamonov) 08A15 | Book |
| 74 | **MR1673047 (2000d:00001)** Smith, Jonathan D. H.; Romanowska, Anna B. Post-modern algebra. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. *John Wiley & Sons, Inc., New York,* 1999. xii+370 pp. ISBN: 0-471-12738-8 (Reviewer: Edgar G. Goodaire) 00A05 (08-01 18-01 20-01) | Book |
| 68 | **MR1932199 (2003i:08001)** Romanowska, Anna B.; Smith, Jonathan D. H. Modes. *World Scientific Publishing Co., Inc., River Edge, NJ,* 2002. xii+623 pp. ISBN: 981-02-4942-X (Reviewer: Ewa Graczyńska) 08-02 (08A05 08B05 08C15 18B99) | Book |
| 57 | **MR2268350 (2008a:20104)** Smith, Jonathan D. H. An introduction to quasigroups and their representations. Studies in Advanced Mathematics. *Chapman & Hall/CRC, Boca Raton, FL,* 2007. xii+340 pp. ISBN: 978-1-58488-537-5; 1-58488-537-8 (Reviewer: Victor T. Markov) 20N05 | Book |
| 49 | **MR0788695 (86k:08001)** Romanowska, A. B.; Smith, J. D. H. Modal theory: an algebraic approach to order, geometry, and convexity. Research and Exposition in Mathematics, 9. *Heldermann Verlag, Berlin,* 1985. xii+158 pp. ISBN: 3-88538-209-1 (Reviewer: Boris M. Schein) 08-02 (06A12 20N15) | Book |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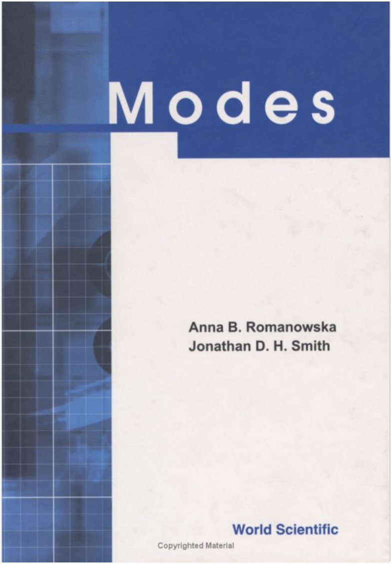
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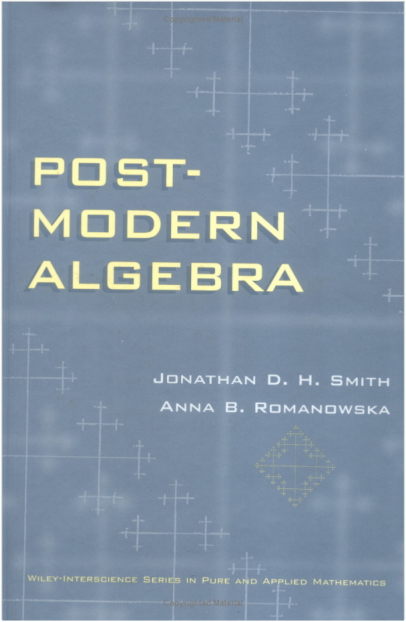
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# Malcev Operation

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In quasigroups: $P(x,y,z)=(x/(y\backslash y))\cdot(y\backslash z)$.

**Lemma**

*Let $Q$ be a quasigroup and $V$ a subquasigroup of $Q\times Q$ containing the diagonal $\hat{Q}=\{(x,x):x\in Q\}$. Then $V$ is a congruence on $Q$.*

**Proof.**

Symmetry:

$$(x,x),\ (x,y),\ (y,y)\in V \Rightarrow (y,x)=(P(x,x,y),P(x,y,y))\in V.$$

Etc. $\square$

# Central congruences

A congruence $V$ on $Q$ is *central* if $\hat{Q}$ is a normal subalgebra of $V$.

**Theorem**

*A quasigroup $Q$ has a unique maximal central congruence, the center congruence $\zeta(Q)$.*

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*Central nilpotence* for quasigroups is now defined iteratively by factoring out the center congruence in every step.

# $\mathcal{Z}$-quasigroups

A quasigroup $Q$ is a $\mathcal{Z}$-*quasigroup* if $\zeta(Q) = Q \times Q$ and $\hat{Q}$ is normal in $Q \times Q$. (Nilpotent of class $\leq 1$.)

# $\mathcal{Z}$-quasigroups

A quasigroup $Q$ is a $\mathcal{Z}$*-quasigroup* if $\zeta(Q) = Q \times Q$ and $\hat{Q}$ is normal in $Q \times Q$. (Nilpotent of class $\leq 1$.)

Note: $\mathcal{Z}$-loops are precisely abelian groups.

**Theorem**

*Let $Q$ be a quasigroup of prime order $p$. Then:*

- *$Q$ is $\mathcal{Z}$-quasigroup, or*
- *$\mathrm{Mlt}(Q) \in \{A_Q, S_Q\}$, or*
- *$p = 11$ and $\mathrm{Mlt}(Q) \in \{PSL_2(11), M_{11}\}$, or*
- *$p = 23$ and $\mathrm{Mlt}(Q) = M_{23}$, or*
- *$p = (q^k - 1)/(q - 1)$ for a prime power $q$ and $PSL_k(q) \leq \mathrm{Mlt}(Q) \leq P\Gamma L_k(q)$.*

# Central isotopy

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Central isotopes of abelian groups are abelian groups.

Central isotopes of $\mathcal{Z}$-quasigroups are $\mathcal{Z}$-quasigroups.

# Quasigroup representations

Smith developed three kinds of representation theory for quasigroups:

- permutation representation,
- character theory,
- module theory.

All come with a twist to account for the lack of associativity.

# Approximate symmetry

... an exact symmetry holding at some level of a hierarchical system.

| $Q$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 2 | 5 | 6 | 4 |
| 2 | 3 | 2 | 1 | 6 | 4 | 5 |
| 3 | 2 | 1 | 3 | 4 | 5 | 6 |
| 4 | 4 | 5 | 6 | 1 | 2 | 3 |
| 5 | 5 | 6 | 4 | 2 | 3 | 1 |
| 6 | 6 | 4 | 5 | 3 | 1 | 2 |

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| 4 | 4 | 5 | 6 | 1 | 2 | 3 |
| 5 | 5 | 6 | 4 | 2 | 3 | 1 |
| 6 | 6 | 4 | 5 | 3 | 1 | 2 |

Let $P = \{1\} \leq Q$ and $P \backslash Q = \{\{1\}, \{2,3\}, \{4,5,6\}\} = \{a_1, a_2, a_3\}$ the orbits of $\mathrm{LMlt}_P(Q) = \langle (2,3)(4,5,6) \rangle = \langle L_1 \rangle$.

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For $x \in Q$, let $R(x) = R_{P\backslash Q}(x)$ be the square matrix indexed by $P\backslash Q$ such that $R(x)(a_i, a_j)$ is the probability that $x$ moves a randomly chosen element of $a_i$ to $a_j$.

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| 3 | 2 | 1 | 3 | 4 | 5 | 6 |
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The monoid generated by all $R(x)$ acts on the space
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The monoid generated by all $R(x)$ acts on the space $\{(x_1 a_1, x_2 a_2, x_3 a_3) : x_1 + x_2 + x_3 = 1\}$.
Combining $a_1$, $a_2$ into $a_1 \cup a_2$ yields an exact symmetry.

# Permutation representation

... essentially an abstract version of the iterated function system exhibited on the previous slide. This can be expressed in terms of coalgebras.

# Character tables

$G = \mathrm{Mlt}(Q)$ acts diagonally on $Q \times Q$ by $L_x(a, b) = (xa, xb)$, etc.

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**Class functions** are those mappings $\theta : Q \times Q \to Q \times Q$ with $\theta = \theta^g$ for every $g \in G$, where $\theta^g(x, y) = \theta(g^{-1}x, g^{-1}y)$. These are complex linear combinations of characteristic functions on conjugacy classes.

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The character table is an $s$-tuple of suitably chosen class functions. There are orthogonality relations, etc.

# Modules

Fix an object $Q$ in a category $\mathcal{C}$. The **slice category** $\mathcal{C}/Q$ has as objects the morphisms $p : E \to Q$ from $\mathcal{C}$, and morphisms

$$f : (p_1 : E_1 \to Q) \to (p_2 : E_2 \to Q)$$

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**Theorem**

*Let $\mathcal{C}$ be the variety of groups and $G$ a group. Then there is a one-to-one correspondence between right $G$-modules and "abelian groups" in the slice category $\mathcal{C}/G$.*

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**Theorem**

*Let $\mathcal{C}$ be the variety of groups and $G$ a group. Then there is a one-to-one correspondence between right $G$-modules and "abelian groups" in the slice category $\mathcal{C}/G$.*

Quasigroup $Q$-modules are then defined to be the abelian groups in the slice category $\mathcal{C}/Q$, where $\mathcal{C}$ is the category of quasigroups.

# Barycentric algebras

Let $C$ be a convex set and $p \in (0, 1)$. Define $\underline{p}(x, y) = (1 - p)x + py$.

# Barycentric algebras

Let $C$ be a convex set and $p \in (0,1)$. Define $\underline{p}(x,y) = (1-p)x + py$. Then:

$$\underline{p}(x,x) = x,$$
$$\underline{p}(x,y) = \underline{1-p}(y,x),$$
$$\underline{q}(z,\underline{p}(y,x)) = \underline{q \circ p}(\underline{q/(p \circ q)}(z,y),x),$$

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Meet semilattices with $\underline{p}(x,y) = xy = x \wedge y$ satisfy the same axioms.

Abstractly, we obtain *barycentric algebras*.

# Modes

**Modes** are idempotent and entropic algebras, that is, their operations satisfy

$$\omega(x, x, \cdots, x) = x,$$

and

$$\omega(\omega'(x_{11}, \dots, x_{1n}), \dots, \omega'(x_{m1}, \dots, x_{mn})) \\ = \omega'(\omega(x_{11}, \dots, x_{m1}), \dots, \omega(x_{1n}, \dots, x_{mn})).$$

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Equivalently, all polynomials are homomorphisms.

Barycentric algebras are instances of modes.

Applications include hierarchical statistical mechanics and modeling of complex systems.

# The context for comtrans algebras

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"Lie algebras vs. formal groups" generalizes to
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"Akivis algebras vs. formal (binary) loops."

For $n \geq 3$, a formal $n$-ary loop is described by $\binom{n}{2}$ Akivis algebras and $\binom{n}{3}$ comtrans algebras in the tangent space.

# Comtrans algebras

...algebraic structure on the tangent bundle of the coordinate ternary loop of a 4-web.

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...algebraic structure on the tangent bundle of the coordinate ternary loop of a $4$-web.

A **comtrans algebra** is a vector space $A$ with two trilinear operations $A \times A \times A \to A$, the commutator $[x, y, z]$ and the translator $\langle x, y, z \rangle$, satisfying the following polynomial identities for all $x, y, z \in A$:

- $[x, y, z] + [y, x, z] = 0$,
- $\langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle = 0$ (Jacobi identity),
- $[x, y, z] + [z, y, x] = \langle x, y, z \rangle + \langle z, y, x \rangle$ (comtrans identity).

# Toward hypercomplex algebras

The standard story of algebraic triplets is that William Rowan Hamilton wanted to generalise the geometric view given by the complex plane (the Argand diagram) to three dimensions so that applications in 3-dimensions could benefit from the system of triplets in an analogous way to how the complex numbers give a powerful way of making applications in 2-dimensions. For example in 1842 Hamilton was so preoccupied with the triplets that even his children were aware of it. Every morning they would inquire:-

> *Well, Papa can you multiply triplets?*

but he had to admit that he could still only add and subtract them.

# Well, academic Papa, can you multiply sedecimtuplets?

# Well, academic Papa, can you multiply sedecimtuplets?

Yes, he can!

# Well, academic Papa, can you multiply sedecimtuplets?

Yes, he can!

The sequence of normed real division algebras—real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and Cayley numbers $\mathbb{K}$—exhibits a successive degradation of properties. The complex numbers are no longer ordered, the quaternions no longer commutative, and the Cayley numbers no longer associative. There is a parallel degradation of the properties of the induced multiplications on the corresponding unit spheres. Thus $S^0$ is a cyclic group, $S^1$ is a non-cyclic abelian group, $S^3$ is a non-abelian group, and $S^7$ is a Moufang loop. This degradation, along with results such as Hurwitz' [Hu] on composition algebras and Adams' [Ad] on odd maps, has led to a consensus that the nested sequence of "hypercomplex numbers" $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{K}$ necessarily terminates at the Cayley numbers [Eb, Sect. 10.3.4]. The only work outside the consensus appears to have been that of Pfister [Pf] on the composition of quadratic forms, although this work was done in more of a number-theoretic context without regard for the algebraic properties of the composition.

The purpose of the current paper is to present an algebraic structure (4.2) on a 16-dimensional Euclidean space $\mathbb{S} = \mathbb{K} \oplus \mathbb{K}f$ (of "sedenions"; cf. "quaternions," "octonions"), such that the Euclidean norm is multiplicative (Theorem 4.1) and the Cayley numbers appear as a subalgebra.

# Sedenions

## Theorem (S 1995)

*Let* $\mathbb{K}$ *be the usual Cayley numbers (real octonions) and let* $f$ *be a new unit. Define addition on* $\mathbb{S} = \mathbb{K} + \mathbb{K}f$ *componentwise, and multiplication by*

$$(x+yf)(u+vf) = \begin{cases} xu + vxf, & \text{if } y = 0, \\ (xy \cdot uy^{-1} - y\overline{v}) + (y\overline{u} - vy^{-1} \cdot xy)f, & \text{else.} \end{cases}$$

*Then:*

- $\mathbb{K}$ *embeds into* $\mathbb{S}$ *as a subalgebra,*
- *the left distributive law holds in* $\mathbb{S}$*,*
- *the norm* $|x + yf| = (x\overline{x} + y\overline{y})^{1/2}$ *is multiplicative,*
- *the* $15$*-sphere* $S = \{z \in \mathbb{S} : |z| = 1\}$ *is a left loop, and it is a loop almost everywhere.*


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