TANGENT AND COTANGENT LOOPOIDS

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Loops

- Let us recall that a quasigroup is is an algebraic structure $< G, \cdot >$ with a binary operation (written usually as juxtaposition, $a \cdot b = ab$) such that $r_g : x \mapsto xg$ (the right translation) and $l_g : x \mapsto gx$ (the left translation) are permutations of G, equivalently, in which the equations $ya = b$ and $ax = b$ are soluble uniquely for x and y respectively. If we assume only that left (resp., right) translations are permutations, we speak about a left quasigroup (resp., right quasigroup.
- A left loop is defined to be a left quasigroup with a right identity e, i.e. $xe = x$, while a right loop is a right quasigroup with a left identity, $ex = x$. A loop is a quasigroup with a two-sided identity element, e, $ex = xe = x$. A loop $< G, \cdot, e >$ with identity e is called an inverse loop if to each element a in G there corresponds an element a^{-1} in G such that

$$a^{-1}(ab) = (ba)a^{-1} = b$$

for all $b \in G$.

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Transversals

Example

Let G be a group with the unit e, H be a subgroup, and $S \subset G$ be a left transversal to H in G, i.e. S contains exactly one point from each coset gH in G/H. This means that any element $g \in G$ has a unique decomposition $g = sh$, where $s \in S$ and $h \in H$ and produces an identification $G = S \times H$ of sets. Let $p_S : G \to S$ be the projection on S determined by this identification. If we assume that $e \in S$, then S with the multiplication

$$s \circ s' = p_S(ss')$$

and e as a right unit is a left loop.

We would like to propose a concepts of loopoid, defined as a nonassociative generalization of a groupoid. Note that here and throughout the presentation, by groupoid we understand a Brandt groupoid, i.e. a small category in which every morphism is an isomorphism, and not an object called in algebra also a magma.

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Groupoids

- These are loops which can be considered as nonassociative generalizations of groups. In the case of genuine groupoids, however, the situation is more complicated, because the multiplication is only partially defined, so the axioms of a loop must be reformulated.
- A convenient way is to think about groupoids as being defined exactly like groups but with the difference that all objects/maps in the definition are relations, like it has been done by Zakrzewski. In particular, the unity is a relation $\varepsilon : \{e\} \longrightarrow\!\!\triangleright\ G$, associating to a point e a subset $M = \varepsilon(e) \subset G$, the set of units. Using this idea, we define semiloopoids, as well as more specific objects which we will call loopoids.
- Infinitesimal parts of Lie groupoids are Lie algebroids and the corresponding 'Lie theory' is well established. This can be partially extended to a differential version of the concept of (semi)loopoid, a differential (semi)loopoid. As the infinitesimal version of associativity is the Jacobi identity, the corresponding 'brackets' will not satisfy the latter.

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Semiloopoids

- Note that the term loopoid has appeared already in a paper by Kinyon in a similar context. The motivating example, however, built as an object 'integrating' the Courant bracket on $\mathsf{T}M \oplus_M \mathsf{T}^* M$, uses the group of diffeomorphisms of the manifold M as 'integrating' the Lie algebra of vector fields on M, not the pair groupoid $M \times M$ as 'integrating' the Lie algebroid $\mathsf{T}M$.

Definition

A semiloopoid over a set M is a structure consisting of a set G together with projections $\alpha, \beta : G \to M$ onto a subset $M \subset G$ (set of units) and a multiplication relation $G_3 \subset G \times G \times G$ such that, for each $g \in G$,

$$(\alpha(g), g, g) \in G_3 \quad \text{and} \quad (g, \beta(g), g) \in G_3\,, \tag{1}$$

and the relations $l_g, r_g \subset G \times G$ defined by

$$(h_1, h_2) \in l_g \quad \Leftrightarrow \quad (g, h_1, h_2) \in G_3\,, \tag{2}$$

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Semiloopoids

We can view l_g and r_g as bijections defined on their domains, D^l_g and D^r_g onto their ranges, R^l_g and R^r_g, respectively.

Definition

(alternative) A semiloopoid over a set M is a structure consisting of a set G including M and equipped with

- a partial multiplication $m : G \times G \supset G_2 \to G$, $m(g,h) = gh$, such that, for all $g \in G$,
$$l_g : D^l_g \to R^l_g\,,\ l_g h = gh\,, \tag{4}$$
is a bijection from $D^l_g = \{h \in G \,|\, (g,h) \in G_2\}$ onto $R^l_g = \{gh \,|(g,h) \in G_2\}$, and
$$r_g : D^r_g \to R^r_g\,,\ r_g h = hg\,, \tag{5}$$
is a bijection from $D^r_g = \{h \in G \,|\, (h,g) \in G_2\}$ onto $R^r_g = \{hg \,|(h,g) \in G_2\}$;
- a pair of projections $\alpha, \beta : G \to M$ such that, for all $g \in G$,
$$\alpha(g)g = g\,,\ g\beta(g) = g\,. \tag{6}$$

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Inverse semiloopoids

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A semiloopoid will be called a left inverse semiloopoid if there is a left inversion map $\varepsilon_l : G \to G$ such that for each $(g, h) \in G_2$ also $(\varepsilon_l(g), gh) \in G_2$ and $\varepsilon_l(g)(gh) = h$. A right inverse semiloopoid can be defined analogously.
A semiloopoid will be called an inverse semiloopoid if there is an inversion map $\varepsilon : G \to G$, to be denoted simply by $\varepsilon(g) = g^{-1}$, such that, for each $(g, h), (u, g) \in G_2$, also $(g^{-1}, gh), (ug, g^{-1}) \in G_2$ and

$$g^{-1}(gh) = h\,, \quad (ug)g^{-1} = u\,.$$

In any inverse semiloopoid the following hold true:
$g^{-1}g = \beta(g) = \alpha(g^{-1})\,, \quad gg^{-1} = \alpha(g) = \beta(g^{-1})$, $(g^{-1})^{-1} = g$, $(gh)^{-1} = h^{-1}g^{-1}$. The latter condition means that one side of the equality makes sens if and only if the other makes sense (the elements are composable) and they are equal.

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In any inverse semiloopoid the following hold true:
$g^{-1}g = \beta(g) = \alpha(g^{-1})\,, \quad gg^{-1} = \alpha(g) = \beta(g^{-1})$, $(g^{-1})^{-1} = g$, $(gh)^{-1} = h^{-1}g^{-1}$. The latter condition means that one side of the equality makes sens if and only if the other makes sense (the elements are composable) and they are equal.

Inverse semiloopoids

Definition

A semiloopoid will be called a left inverse semiloopoid if there is a left inversion map $\varepsilon_l : G \to G$ such that for each $(g, h) \in G_2$ also $(\varepsilon_l(g), gh) \in G_2$ and $\varepsilon_l(g)(gh) = h$. A right inverse semiloopoid can be defined analogously.

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Unities associativity

The maps α, β in can be rather pathological. Let us assume now, that a semiloopoid G over M, with a partial multiplication m and projections $\alpha, \beta : G \to M$, satisfies a very weak associativity condition, hereafter called unities associativity:

$$(xy)z = x(yz) \text{ if one of } x, y, z \text{ is a unit (i.e. belongs to } M). \tag{7}$$

The following proposition shows that the condition of unities associativity for a semiloopoid over M is rather strong and implies that the anchor map $(\alpha, \beta) : G \to M \times M$ has nice properties, similar to these for groupoids.

Proposition

A semiloopoid G over M satisfies the unities associativity condition if and only if

$$G_2 = \{(g, h) \in G \times G \mid \beta(g) = \alpha(h)\} \tag{8}$$

and

$$(\alpha, \beta) : G \to M \times M \tag{9}$$

is a semiloopoid morphism into the pair groupoid $M \times M$, i.e.

$$\alpha(gh) = \alpha(g) \text{ and } \beta(gh) = \beta(h)\,. \tag{10}$$

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Loopoids

The unities associativity assumption implies that each element g of G determines the left and right translation maps are injective:

$$l_g : \mathcal{F}^\alpha(\beta(g)) \to \mathcal{F}^\alpha(\alpha(g))\,, \quad r_g : \mathcal{F}^\beta(\alpha(g)) \to \mathcal{F}^\beta(\beta(g))\,. \qquad (11)$$

Definition

A semiloopoid satisfying the unities associativity assumption and such that the maps (11) are bijective will be called a loopoid.

In a loop, the multiplication is globally defined, so the unity associativity is always satisfied by properties of the unity element. In this sense, loops are loopoids over one point.

Proposition

Let G be a loopoid over M with the source and target maps $\alpha, \beta : G \to M$. Then, for each $u \in M$, the multiplication in G induces on the set

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Differential loopoids

- A loopoid G over a set M will be denoted simply by the symbol $G \rightrightarrows M$.
- We will consider differential (smooth) loopoids. The inverse loopoid $G \rightrightarrows M$ is said to be a differential if G and M are smooth manifolds and all the structural maps are smooth with α and β being smooth submersions.
- If $G \rightrightarrows M$ is a differential inverse loopoid then m is a submersion, $\iota : M \to G$ is an injective immersion and the inverse is a diffeomorphism. Also left translations and right translations are diffeomorphisms of the corresponding α- and β-fibers.
- Instead of differential inverse loopoid we can consider also weaker concepts of a (differential) left inverse loopoid and right inverse loopoid. In these cases we have not the inverse map $\varepsilon : G \to G$, but two inverse maps $\varepsilon_l, i_r : G \to G$, the left inverse $\varepsilon_l(g) = g_l^{-1}$ and the right inverse $\varepsilon_r(g) = g_r^{-1}$, and we assume

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Tangent and cotangent loopoid

Theorem

If $G \rightrightarrows M$ is a differential (left inverse, inverse) loopoid, then TG and T^*G carry canonical structures of (left inverse, inverse) differential loopoids over TM and $A^*G = \nu^*(G, M)$, respectively. Here, $AG = \nu(G, M)$ is the normal bundle to the submanifold $M \subset G$.

- The tangent loopoid $\mathsf{T}G$ is obtained just by applying the tangent functor: the source and target maps are $\mathsf{T}\alpha$ and $\mathsf{T}\beta$, the partial multiplication is $\mathsf{T}m$, $\mathsf{T}m(X, X') = X \bullet X'$, etc.
- In the cotangent loopoid T^*G, the source mapping $\tilde{\alpha} : \mathsf{T}^*G \to A^*G$ is defined as follows. Let μ be a cotangent vector to G at the element g. We restrict μ to $T_g\mathcal{F}^\beta$ and then pull back by r_g to move it to $T_{\alpha(g)}\mathcal{F}^\beta$. Finally, we identify the tangent space $T_{\alpha(g)}\mathcal{F}^\beta$ with the conormal space to M, since the β-fibre is transverse to M.
- By interchanging α and β and "right" and "left", we construct in a similar way the target map $\tilde{\beta}$. Finally, $\gamma\gamma'(X \bullet X') = \gamma(X) + \gamma'(X')$.

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Example

- Consider on $G = \mathbb{R}$ a differential loop structure with the multiplication
$$x \circ y = x + 2y\,.$$
- On $\mathsf{T}G = \mathsf{T}\mathbb{R} = \{(x, \dot{x}) : x, \dot{x} \in \mathbb{R}\}$ we have the tangent loop structure
$$(x, \dot{x}) \bullet (y, \dot{y}) = (x + 2y, \dot{x} + 2\dot{y})$$
which is a differential loopoid structure over $M = \{(0, 0)\}$.
- On $\mathsf{T}^*G = \mathsf{T}^*\mathbb{R} = \{(x, p) :\ x, p \in \mathbb{R}\}$ we have a differential loopoid structure over $M = A^*G = \mathsf{T}^*_0\mathbb{R} = \mathbb{R}^* = \{p\ : p \in \mathbb{R}\}$.
- The target and the source projections are
$$\tilde{\alpha}(x, p) = \frac{1}{2}p\,,\ \ \tilde{\beta}(x, p) = p\,.$$
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- On $\mathsf{T}G = \mathsf{T}\mathbb{R} = \{(x, \dot{x}) : x, \dot{x} \in \mathbb{R}\}$ we have the tangent loop structure
$$(x, \dot{x}) \bullet (y, \dot{y}) = (x + 2y, \dot{x} + 2\dot{y})$$
which is a differential loopoid structure over $M = \{(0, 0)\}$.
- On $\mathsf{T}^*G = \mathsf{T}^*\mathbb{R} = \{(x, p) : x, p \in \mathbb{R}\}$ we have a differential loopoid structure over $M = A^*G = \mathsf{T}^*_0\mathbb{R} = \mathbb{R}^* = \{p : p \in \mathbb{R}\}$.
- The target and the source projections are
$$\tilde{\alpha}(x, p) = \frac{1}{2}p\,,\ \tilde{\beta}(x, p) = p\,.$$
- The partial product is defined by
$$(x, \frac{1}{2}p)(y, p) = (x + 2y, p)\,.$$

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THANK YOU FOR YOUR ATTENTION!