ESSENTIALLY EQUIVALENCE OF HYPERIDENTITIES IN SEMIGROUPS

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The balanced identities in quasigroups were for the first time studied by Belousov. The classification of hyperidentities of associativity in invertible algebras was given by Belousov. Moreover, their classification in q-algebras and e-algebras (with more than one operation) was given by Yu. Movsisyan, the only relevant forms are as follows

$$X(x, Y(y, z)) = Y(X(x, y), z)$$
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Definition of q-algebra and e-algebra

- The binary algebra (Q, Σ) is called q-algebra, if $Q(A)$ is a quasigroup for some operation $A \in \Sigma$.
- The binary algebra (Q, Σ) is called e-algebra, if $Q(A)$ has a unit for some operation $A \in \Sigma$.

The semigroup $Q(\cdot)$ is called hyperassociative, if it polynomially satisfies the following hyperidentity of associativity:

$$X(x, X(y, z)) = X(X(x, y), z)$$

The variety of all hyperassociative semigroups is defined by the finite system of identities. By Denecke, Koppitz and Paseman the finite basis of identities consists of about 1000 identities.

In works of Polak a basis of identities of this variety which consists only of four identities is described

Theorem

The semigroup $Q(\cdot)$ is hyperassociative iff the following identies are valid in $Q(\cdot)$:

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\begin{aligned}
x^4 &= x^2 \\
xyxzxyx &= xyzyx \\
xy^2z^2 &= xyz^2yz^2 \\
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Definition of hyperidentity

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The following second order formula is called a hyperidentity:

$$\forall X_1, \ldots, X_m; \forall x_1, \ldots, x_n \ (W_1 = W_2)$$

where $X_1, \ldots, X_m$ are functional variables, and $x_1, \ldots, x_n$ are subject variables in words W_1, W_2. Usually, a hyperidentity is specified without prefix of the universal quantifiers:

$$W_1 = W_2$$

- The numbers m and n in the hyperidentity are called the functional and object rank, respectively
- A hyperidentity is said to be **non-trivial** if its functional rank is greater than 1, i.e. if $m > 1$
- A hyperidentity is said to be **trivial** otherwise, i.e. if $m = 1$

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- A hyperidentity is said to be $n-$ary if its functional variables are $n-$ary
- The $n-$ary hyperidentity is called **unary**, **binary**, **ternary** for $n = 1, 2, 3$
- If the arities of the functional variables are:

$$|X_1| = n_1, \ldots, |X_m| = n_m$$

  then the hyperidentity $W_1 = W_2$ is called $\{n_1, \ldots, n_m\}$-hyperidentity.
- According to the definition, the hyperidentity $W_1 = W_2$ is said to be satisfied in the algebra (Q, Σ) if this equality holds when every functional variable X_i is replaced by any arbitrary operation of the corresponding arity from Σ and every object variable x_j is replaced by an arbitrary element from Q .

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Other definitions

- A hyperidentity is **balanced** if each object variable of the hyperidentity occurs in both parts of the equality $W_1 = W_2$ only once.
- A balanced hyperidentity is called **first sort** hyperidentity, if the object variables on the left and right parts of the equality are ordered identically.
- The number of the object variables in a balanced hyperidentity is called **length** of this hyperidentity.

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The definition of binary terms in a semigroup $Q(\cdot)$

If $Q(\cdot)$ is a semigroup, then the following function is called a *binary polynomial* (*term*) of a semigroup $Q(\cdot)$:

$$F(x, y) = t_1^{\varepsilon_1} t_2^{\varepsilon_2} \dots t_n^{\varepsilon_n}$$

where

$$n \in N, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in N, \quad t_1, t_2, \dots, t_n \in \{x, y\}, t_i \neq t_{i+1}$$

The number n is called the length of this representation of the polynomial $F(x, y)$. However, due to the identities in the semigroup $Q(\cdot)$, identical polynomials can have different representations of this form.

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The number n is called the length of this representation of the polynomial $F(x,y)$. However, due to the identities in the semigroup $Q(\cdot)$, identical polynomials can have different representations of this form.

Let the set of all binary polynomials of a semigroup $Q(\cdot)$ be Q^2_{pol}. We say that the binary hyperidentity $W_1 = W_2$ is satisfied polynomially in a semigroup $Q(\cdot)$, if this hyperidentity is satisfied in the binary algebra (Q, Q^2_{pol}).

The definition of binary essential polynomials in a semigroup $Q(\cdot)$

Definition

The polynomial $F(x, y)$ depends on the variable x essentially in the semigroup $Q(\cdot)$ if there are elements $x_1, x_2 \in Q$ such that

$$F(x_1, y) \neq F(x_2, y)$$

In the same way the essentially dependence of the polynomial $F(x, y)$ on the variable y is defined.
The polynomial $F(x, y)$ is called essential if it depends on both variables x and y essentially.

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Essential hyperidentities of semigroups

Let us denote Q^2_{epol} the set of all binary essential polynomials of the semigroup $Q(\cdot)$.

Definition

We say that the binary hyperidentity $W_1 = W_2$ is essentially satisfied (valid) or is satisfied for essential polynomials in the semigroup $Q(\cdot)$, if this hyperidentity is satisfied in the binary algebra (Q, Q^2_{epol}).

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Equivalence of hyperidentities

Definition

We say that two hyperidentities are equivalent (written as $\Leftrightarrow$), if they simultaneously are either polynomially satisfied or none of them is polynomially satisfied in any semigroup $Q(\cdot)$. It is said the hyperidentity (h_1) implies the hyperidentity (h_2), written as $(h_1) \Rightarrow (h_2)$, if in all semigroups where the hyperidentity (h_1) is satisfied polynomially, the hyperidentity (h_2) is also satisfied polynomially.

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Essential equivalence of hyperidentities in semigroups

Theorem

Any non-trivial associative essential hyperidentity is essentially equivalent to one of the following hyperidentities:

$$X(X(x,y),z) = X(x,Y(y,z)) \tag{1}$$
$$X(X(x,y),z) = Y(x,X(y,z)) \tag{2}$$
$$X(X(x,y),z) = Y(x,Y(y,z)) \tag{3}$$
$$X(Y(x,y),z) = X(x,Y(y,z)) \tag{4}$$
$$X(Y(x,y),z) = Y(x,X(y,z)) \tag{5}$$

Moreover, we have the following implications: (1) $\Rightarrow_e$ (3), (1) $\Rightarrow_e$ (2) $\Rightarrow_e$ (5) $\Rightarrow_e$ (4).

Essential hyperidentity (1) in semigroups

Theorem

The hyperidentity

$$X(X(x,y),z) = X(x,Y(y,z))$$

is essentially satisfied in the semigroup $Q(\cdot)$ if and only if this semigroup satisfies one of the following systems of identities:

$$\begin{cases} xyz = xzy \\ xyz = zxy \\ x^2y = x^3 \end{cases}$$

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Essential hyperidentity (2) in semigroups

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Essential hyperidentity (4) in semigroups

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Essential hyperidentity (4) in semigroups

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Essential hyperidentity (4) in semigroups

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Essential hyperidentity (5) in semigroups

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$$X(Y(x,y),z) = Y(x,X(y,z))$$

is essentially satisfied in the semigroup $Q(\cdot)$ if and only if this semigroup satisfies one of the following systems of identities:

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Essential hyperidentity (5) in semigroups

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Essential hyperidentity (5) in semigroups

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