Affine Steiner loops

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Steiner triple systems and loops

Definition

A Steiner triple system STS is an incidence structure consisting of points and blocks such that every two distinct points are contained in precisely one block and any block has precisely three points.
A loop $(L, \cdot)$ *is a quasigroup which has identity element e. This means the left and right translations* $\lambda_a : y \mapsto a \cdot y : L \to L$ *and* $\rho_a : y \mapsto y \cdot a : L \to L$, $a \in L$ *are permutations of L. The multiplication table for a finite loop is a Latin square.*

The classical examples for Steiner triple system are the lines in an affine geometry $AG(d, 3)$ over $GF(3)$, for $d = 2$ this gives the unique $STS(9)$ and the lines of a projective geometry $PG(d, 2)$ over $GF(2)$ which yields for $d = 2$ the unique $STS(7)$ which is also called the Fano plane.

Definition

Let $\mathcal{S}$ be a Steiner triple system, and let $\Omega \in \mathcal{S}$ be fixed. For each $x \in \mathcal{S}$, define its opposite $-x$ as the third point $\mu(x)$ in the triple $\{x, \Omega, \mu(x)\}$ through x and Ω, define the addition $x + \Omega = x$, $x + x = -x$, and, for any $y \neq x \in \mathcal{S} \setminus \{\Omega\}$,

$$x + y = -z,$$

where z is the third point in the triple through x, y.

This definition gives a group precisely if the *STS* is an affine geometry over $GF(3)$.
The triples $\{x, y, z\}$ of the *STS* corresponding to an affine Steiner loop are characterized by the property that

$$x + y + z = \Omega.$$

Hence for any triple in the *STS* the associative property holds.

A. Caggegi, G. Falcone, M. Pavone, *On the additivity of block designs* J. Algebr. Comb. 45 (1), 271-294.

Since for every $x \in STS$ one has $x + x + x = \Omega$ an affine Steiner loop has exponent 3.
An affine Steiner loop satisfies the weak inverse property.
O. Chein, Examples and methods of construction, II.9.9 Example in O. Chein, H. O. Pflugfelder, J. D. H. Smith, eds. Quasigroups and Loops: Theory and Applications. Heldermann (1990), p. 86.

A STS is called a Hall triple system HTS if any three elements, not in a triple, are contained in an affine plane $AG(2,3)$. For Hall triple systems the above definition yields commutative Moufang loop of exponent 3. General an affine Steiner loop only fulfill

$$x + \big((x + y) + (x + y)\big) = y + y,$$

M. H. Armanious, *Commutative Loops of Exponent 3 with* $x \cdot (x \cdot y)^2 = y^2$, Demonstratio Math. 35 (3), 469–475.

Conversely, if $\mathcal{L}_{\mathcal{S}}$ is a commutative loop such that $3x = \Omega$ and fulfilling the weak inverse property, then $\mathcal{L}_{\mathcal{S}}$ gives the structure of a Steiner triple system $\mathcal{S}_{\mathcal{L}}$.

Using the commutativity for any $a \in \mathcal{L}_{\mathcal{S}}$ the left translation map λ_a coincides with the right translation map ρ_a.

If $L_1(\circ)$ and $L_2(*)$ are loops, a triple of bijections $(\alpha, \beta, \gamma) : L_1 \longrightarrow L_2$ such that $\alpha(x) * \beta(y) = \gamma(x \circ y)$ is called an *isotopism*.

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two affine Steiner loops associated to the same STS by fixing two different elements Ω_1 and Ω_2, and if we denote the opposite maps by μ_1 and μ_2, according to the triples $\{x, \Omega_1, \mu_1(x)\}$ and $\{x, \Omega_2, \mu_2(x)\}$ in $\mathcal{T}$, then the map $\gamma : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$, $\gamma(x) = \mu_2(\mu_1^{-1}(x))$ induces an isotopism $(\mathrm{id}, \mathrm{id}, \gamma) : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$.

Definition

The kernel of a homomorphism $\alpha : (L, \circ) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L. In case of a commutative loop L a subloop N of L is normal if $x + (y + N) = (x + y) + N$ holds for all $x, y \in L$.

Theorem

Let S be a Steiner triple system and L_S the corresponding affine Steiner loop.

i) *L_R is a subloop of L_S if and only if R is a Steiner triple subsystem of S containing Ω.*

ii) *If L_R is a normal subloop of L_S then each coset $x + L_R$ corresponds to a subsystem of S.*

iii) *If L_R is a normal subloop of L_S then the factor loop Q yields a Steiner triple system S_Q.*

In the case where L_R is a normal subloop of L_S, the corresponding left cosets are not necessarily isomorphic subsystems.

Theorem

Let $\mathcal{S}$ be a Steiner triple system and $\mathcal{L}_\mathcal{S}$ the corresponding affine Steiner loop with identity element Ω. The automorphisms of $\mathcal{S}$ fixing Ω are exactly the automorphisms of $\mathcal{L}_\mathcal{S}$.

Corollary

Let f_0 be an automorphism of $\mathcal{L}_\mathcal{S}$ fixing the identity element Ω. The map $f(x) = f_0(x) + a$ is an isomorphism of $\mathcal{L}_\mathcal{S}$ onto the isotopic affine Steiner loop $\mathcal{L}_\mathcal{S}{}'$ corresponding to $\mathcal{S}$ with identity element $a = f(\Omega)$. Moreover, the map f yields an automorphism of $\mathcal{S}$ if and only if the translation $\rho_a : x \mapsto x + a$ is also an automorphism of $\mathcal{S}$.
If $\mathcal{S}$ is a Hall triple system, then the translations ρ_b are automorphisms of $\mathcal{S}$ and the automorphisms of $\mathcal{S}$ have the form $f(x) = f_0(x) + a$, where f_0 is an automorphism of $\mathcal{L}_\mathcal{S}$ and $a = f(\Omega)$.

Theorem

Let $\mathcal{S}$ be a Steiner triple system with n points, and let $\mathcal{L}_\mathcal{S}$ be the associated affine Steiner loop with identity element $\Omega \in \mathcal{S}$. Then each translation of $\mathcal{L}_\mathcal{S}$ is even and has the form

$$\lambda_x = (\Omega, x, -x)\tau_1 \dots \tau_r$$

where each τ_j is an even permutation of the form

$$\tau_j = (v_1, v_2, ..., v_j)(-v_j, ..., -v_2, -v_1).$$

Thus the multiplication group $\mathrm{Mult}(\mathcal{L}_\mathcal{S})$ generated by all translations of the loop $\mathcal{L}_\mathcal{S}$ is contained in A_n, and the stabilizer $\mathrm{Stab}_{\mathrm{Mult}(\mathcal{L}_\mathcal{S})}(\Omega)$ is contained in A_{n-1}.

Proof

Each λ_x can be written as $\lambda_x = (\Omega, x, -x)\sigma_x$, where σ_x is a permutation without any fixed point on the set $\mathcal{L}_\mathcal{S} \setminus \{\Omega, x, -x\}$ for each $x \in \mathcal{L}_\mathcal{S} \setminus \{\Omega\}$.
We fix an element $x \neq \{\Omega\}$. As σ_x has no fixed point on the set $\mathcal{L}_\mathcal{S} \setminus \{\Omega, x, -x\}$ it does not contain any 1-cycle. If σ_x contains the j-cycle $(v_1, v_2, v_3, ..., v_j)$-cycle, then the STS has the following blocks: $\{x, v_1, -v_2\}$, $\{-v_2, \Omega, v_2\}$, $\{x, v_2, -v_3\}$, $\{-v_3, \Omega, v_3\}$, $\{x, v_3, -v_4\}$, $\{-v_4, \Omega, v_4\}$, $\dots$, $\{x, v_{j-1}, -v_j\}$, $\{-v_j, \Omega, v_j\}$, $\{x, v_j, -v_1\}$, $\{-v_1, \Omega, v_1\}$, since $x + v_1 = v_2$, $x + v_2 = v_3$, $\dots$, $x + v_{j-1} = v_j$, $x + v_j = v_1$. Using these blocks the permutation σ_x has also the j-cycle $(-v_2, -v_1, -v_j, ..., -v_3)$ containing precisely the elements $-v_1, -v_2, ..., -v_j$. So each j-cycle $(v_1, v_2, v_3, ..., v_j)$ of σ_x appears in σ_x together with j-cycle $(-v_j, ..., -v_3, -v_2, -v_1)$.
Since in σ_x every j-cycle appears with such a disjoint j-cycle, it is an even permutation. Then also every $\lambda_x = (\Omega, x, -x)\sigma_x$ is an even permutation.

Steiner triple systems are often studied through their *configurations*, which are given subsets of triples. Starting from two triples $\{z, a_1, a_2\}$ and $\{z, b_1, b_2\}$ through one point z, two cases are possible: either the third point c_1 in the triple through a_1 and b_1 coincides with the third point c_2 in the triple through a_2 and b_2 (Pasch configuration), or not. In the latter case, one can distinguish further the case where the triple containing the two points c_1 and c_2 contains also z (the mitre configuration centered in z).

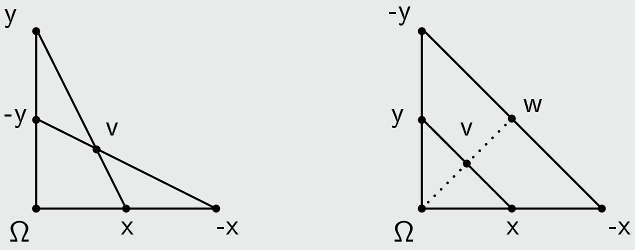

Figure: A Pasch configuration (left) and a mitre (right) centered in Ω

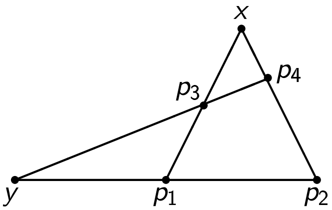

Figure: Veblen point x

Definition

Let x be a point in a Steiner triple system, and let $y \neq x$. If, together with $\{x, p_1, p_3\}$, $\{x, p_2, p_4\}$, and $\{y, p_1, p_2\}$, also $\{y, p_3, p_4\}$ is a triple of $\mathcal{S}$, then x is called a Veblen point. Alternatively, the point x is called a Veblen point, if any two triples through x determine a Fano plane.

O. Veblen and John Wesley Young, *Projective geometry*. Vol. 2. Ginn, (1918) and Ch. J. Colbourn, A. Rosa, *Triple Systems*, Oxford mathematical monographs, Clarendon Press, 1999.

Definition

Let z be a point in a Steiner triple system, and let $u \neq z$. If, together with $\{z, p_1, p_3\}$, $\{z, p_2, p_4\}$, $\{u, p_1, p_2\}$and $\{w, p_3, p_4\}$, also $\{u, w, z\}$ is a triple of $\mathcal{S}$, then z is called a Hall point. Alternatively, the point z is called a Hall point, if any two triples through z determine a mitre centered in z.

For any $z \in \mathcal{L}_{\mathcal{S}}$, we will use the following map

$$\iota_z = \rho_{-z}\rho_z = \lambda_{-z}\lambda_z$$

which plays a role in distinguishing the case where Ω is contained in a Pasch configuration from the case where Ω is a Hall point.

Theorem

1.i) $\{-x, \Omega, x\}$ *and* $\{-y, \Omega, y\}$ *define a mitre centered in* Ω *if and only if* $x + y = -(-x - y)$ *and* $\iota_x(y) = y$*,*

1.ii) Ω *is a Hall point if and only if* $\iota_x(y) = (y + x) - x = y$ *for all* x *and* y *in* $\mathcal{S}$ *equivalently,* $\mathcal{L}_{\mathcal{S}}$ *has the inverse property,*

1.iii) *Any point in* $\mathcal{S}$ *is a Hall point if and only if* $\mathcal{S}$ *is a HTS (see D. Král, E. Máčajová, A. Pór, J.-S. Sereni Characterization results for Steiner triple systems and their application to edge-colorings of cubic graphs Canad. J. Math., 62 (2) , pp. 355-381).*

2.i) $\{-x, \Omega, x\}$ *and* $\{-y, \Omega, y\}$ *define a Pasch configuration if and only if* $x + y = -x - y$*. Moreover, in this case* $\iota_x \neq \mathrm{id}$*.*

2.ii) Ω *is a Veblen point if and only if for all* $y \neq x$ *it holds* $x + y = -x - y$ *and* $\iota_x(\pm y) = \mp y$*,*

2.iii) *Any point in* $\mathcal{S}$ *is a Veblen point if and only if* $\mathcal{S}$ *is a projective geometry* $\mathrm{PG}(d, 2)$ *(cf. Th. 8.15, p. 147, in Ch. J. Colbourn, A. Rosa, Triple Systems, Oxford mathematical monographs).*

Theorem

Let $\mathcal{S}$ be a $\mathrm{STS}(n)$ containing a Veblen point V. Then $n \equiv 3 \pmod 4$ and it holds

i) *V is the neutral element Ω of $\mathcal{L}_{\mathcal{S}}$ if and only if, for any $x \neq \Omega$,*

$$\rho_x = (\Omega, x, -x)(p_1, q_1)(-q_1, -p_1) \cdots (p_{\frac{n-3}{4}}, q_{\frac{n-3}{4}})(-q_{\frac{n-3}{4}}, -p_{\frac{n-3}{4}}).$$

Moreover, $\rho_{-x} =$

$$(\Omega, -x, x)(p_1, -q_1)(q_1, -p_1) \cdots (p_{\frac{n-3}{4}}, -q_{\frac{n-3}{4}})(q_{\frac{n-3}{4}}, -p_{\frac{n-3}{4}}),$$

whereas ρ_{p_i} interchanges x with q_i, and $-x$ with $-q_i$.

ii) *If $V \neq \Omega$, then*

$$\rho_V = (\Omega, x, -x)(p_1, q_1)(-q_1, -p_1) \cdots (p_{\frac{n-3}{4}}, q_{\frac{n-3}{4}})(-q_{\frac{n-3}{4}}, -p_{\frac{n-3}{4}})$$

$$\rho_{-V} = (\Omega, -x, x)(p_1, -q_1)(q_1, -p_1) \cdots (p_{\frac{n-3}{4}}, -q_{\frac{n-3}{4}})(q_{\frac{n-3}{4}}, -p_{\frac{n-3}{4}}).$$

If $\mathcal{R} \leq \mathcal{S}$ is a Steiner triple subsystem of order 9 containing the triple $\{-x, \Omega, x\}$, then the restriction of ρ_x to $\mathcal{R}$ is simply $\rho_x = (\Omega, x, -x)(y, -v, u)(-u, v, -y)$.

Corollary

If $\mathcal{S}$ is a Hall triple system, then $x \mapsto -x$ is an automorphism of $\mathcal{L}_\mathcal{S}$. Moreover, $n = 3^k$ and it holds

$$\rho_x = (\Omega, x, -x)(a_1, b_1, c_1)(-c_1, -b_1, -a_1) \cdots (a_t, b_t, c_t)(-c_t, -b_t, -a_t).$$

The investigation of the structure of any affine Steiner loop can be reduced to consecutive extensions of simple ones. Hence we studied extensions of a normal affine Steiner loop N by an affine Steiner loop Q which yields affine Steiner loop L.

Extensions of normal (sub-)loops N by (factor) loops Q are much more relaxed than in the case of groups
(cf. P. T. Nagy: Nuclear properties of loop extensions, Results in Math. (2019), 74: 100).

Definition

Let N and Q be affine Steiner loops of order w and z, respectively, and let $\mathcal{Q}(N)$ be the set of $w \times w$ latin squares with coefficients in the set N.
An operator $\Phi : Q \times Q \longrightarrow \mathcal{Q}(N)$, which maps the pair $(\bar{x}, \bar{y})$ to a latin square $\Phi_{\bar{x},\bar{y}} : N \times N \longrightarrow N$, and fulfills the following conditions for all $(\bar{x}, x'), (\bar{y}, y'), (\bar{z}, z') \in Q \times N$:

- i) $\Phi_{\bar{y},\bar{x}}(y', x') = \Phi_{\bar{x},\bar{y}}(x', y')$, *that is,* $\Phi_{\bar{y},\bar{x}}$ *is the transpose of* $\Phi_{\bar{x},\bar{y}}$*;*
- ii) *the (symmetric) latin square* $\Phi_{\bar{0},\bar{0}}$ *is the table of addition of* N*;*
- iii) $\Phi_{\bar{x},\bar{0}}(x', 0') = x'$
- iv) $\Phi_{\bar{x},-\bar{x}}(x', \Phi_{\bar{y},\bar{z}}(y', z')) = 0$ *if and only if* $\Phi_{-\bar{z},\bar{z}}(\Phi_{\bar{x},\bar{y}}(x', y'), z') = 0$*;*
- v) $\Phi_{\bar{x},-\bar{x}}(x', \Phi_{\bar{x},\bar{x}}(x', x')) = 0$*;*

is called a Steiner operator.

Theorem

Let N and Q be affine Steiner loops of order w and z, respectively, and let $\Phi : Q \times Q \longrightarrow \mathcal{Q}(N)$ *be a Steiner operator.*
If we define on $L = Q \times N$ *the addition*

$$(\bar{x}, x') + (\bar{y}, y') = (\bar{x} + \bar{y}, \Phi_{\bar{x},\bar{y}}(x', y')),$$

then L is an affine Steiner loop of order $v = wz$*, having N as a normal subloop and such that* L/N *is isomorphic to Q.*

Conversely, any affine Steiner loop L, having a normal subloop N and a factor loop $Q = L/N$ *is isomorphic, for some given Steiner operator* Φ*, to the above one.*

Affine Hyperplane

Theorem

$\mathcal{S}_0$ is an affine hyperplane of $\mathcal{S}$ containing Ω, if and only if $\mathcal{L}_{\mathcal{S}_0}$ is a normal subloop of index 3 in $\mathcal{L}_{\mathcal{S}}$.

An *affine hyperplane* of $\mathcal{S}$ is a subsystem $\mathcal{S}_0$, such that for any $x \notin \mathcal{S}_0$ the set $\mathcal{S}_1$ of blocks through x which do not intersect $\mathcal{S}_0$ turns out to be a second subsystem. Hence $\mathcal{S}_0$ is an affine hyperplane of $\mathcal{S}$, if, and only if, $\mathcal{S}$ is the union of three pairwise disjoint subsystems

$$\mathcal{S} = \mathcal{S}_{-1} \cup \mathcal{S}_0 \cup \mathcal{S}_1,$$

of the same cardinality $w = \frac{n}{3}$, hence it is necessary that
$n \equiv 3 \bmod 6$
(cf. J. Doyen, X. Hubaut, M. Vandensavel, *Ranks of Incidence Matrices of Steiner Triple Systems.* Mathematische Zeitschrift 163: 251-260.)

Schreier Extension

Assume that $\mathcal{L}_\mathcal{S}$ is the Schreier extension of the normal subgroup $\mathcal{N}$ by an affine Steiner loop $\mathcal{L}_\mathcal{K}$
(cf. P. T. Nagy and K. Strambach, *Schreier Loops*, Czechoslovak Math. J. 58 (133), 759–786).
Since the affine loop $\mathcal{L}_\mathcal{S}$ is abelian, this extension is central and it is realized on $\mathcal{L}_\mathcal{K} \times \mathcal{N}$ by the multiplication

$$(1) \qquad (\kappa_1, n_1) \circ (\kappa_2, n_2) = \big(\kappa_1 + \kappa_2, n_1 + n_2 + f(\kappa_1, \kappa_2)\big),$$

where $f : \mathcal{L}_\mathcal{K} \times \mathcal{L}_\mathcal{K} \to \mathcal{N}$ is a function with the property $f(\kappa_1, \kappa_2) = f(\kappa_2, \kappa_1)$ for all $\kappa_1, \kappa_2 \in \mathcal{L}_\mathcal{K}$ satisfying $f(0, \kappa_2) = f(\kappa_1, 0) = \Omega$.

Simple affine Steiner loops

If $\mathcal{L}_\mathcal{S}$ is simple, i.e. it has no proper normal subloop, then the group $\mathrm{Mult}(\mathcal{L}_\mathcal{S})$ is primitive (see A. A. Albert, *Quasigroup I*, Trans. Amer. Math. Soc. 54 (3), 507-519, Th. 8, p. 516).
Write $\sigma_x = \lambda_x(\Omega, x, -x)^{-1}$, $x \in \mathcal{L}_\mathcal{S} \setminus \{\Omega\}$.

Theorem

Let $\mathcal{S}$ be a simple Steiner triple system with $n > 3$ points and $\mathcal{L}_\mathcal{S}$ the corresponding affine Steiner loop.
(i) The group $\mathrm{Mult}(\mathcal{L}_\mathcal{S})$ of $\mathcal{L}_\mathcal{S}$ is isomorphic to A_n, if and only if $\mathrm{Mult}(\mathcal{L}_\mathcal{S})$ contains one of the permutations σ_x.
(ii) If the order of one of the permutations σ_x is not divisible by 3, then the group $\mathrm{Mult}(\mathcal{L}_\mathcal{S})$ of $\mathcal{L}_\mathcal{S}$ is isomorphic to A_n.

In K. Strambach, I. Stuhl, *Translation groups of Steiner loops.* Discrete Mathematics, 309(13), 4225-4227, it is proved that if the order of any product of two different translations of the STS of size $n > 3$ is odd, then the multiplication group $\mathrm{Mult}(L)$ of the corresponding projective Steiner loop L of order $n+1$ contains the alternating group of order $n+1$.

Proof

If one has $\sigma_x \in \mathrm{Mult}(\mathcal{L}_\mathcal{S})$, then the permutation $\lambda_x \sigma_x^{-1} = (\Omega, x, -x)$ is a 3-cycle in the primitive subgroup $\mathrm{Mult}(\mathcal{L}_\mathcal{S})$ of A_n. By Jordan's theorem on permutations, $\mathrm{Mult}(\mathcal{L}_\mathcal{S}) = A_n$. This proves assertion (i).

(ii) If the order of the permutation σ_x is $3k+2$ for some $x \in \mathcal{L}_\mathcal{S}$, then one has $\sigma_x = \lambda_x^{3k+3}$. If the order of the permutation σ_x is $3k+1$ for some $x \in \mathcal{L}_\mathcal{S}$, then one has $\sigma_x = \lambda_x^{3(2k+1)}$. Hence $\sigma_x \in \mathrm{Mult}(\mathcal{L}_\mathcal{S})$ and we are done by part i).

Theorem

If $\mathcal{S}$ is a simple Steiner triple subsystem of order n containing a Veblen point, then $\mathcal{L}_\mathcal{S}$ is the loop having the alternating group A_n as its multiplication group, and $\mathcal{L}_\mathcal{S} = A_n/A_{n-1}$.

Corollary

If $\mathcal{S}$ is a Steiner triple system of order $n = 13$. Then the group $\mathrm{Mult}(\mathcal{L}_\mathcal{S})$ is the alternating group A_{13}.

This follows from the classification of R. M. Guralnick, *Subgroups of prime power index in a simple group* J. Algebra 81 (2), 304-311.

Proposition

Let $\mathcal{S}$ be a Steiner triple system of prime power cardinality $n \neq \frac{q^a-1}{q-1}$, for any prime power q. If $\mathrm{Mult}(\mathcal{L}_\mathcal{S})$ is simple, then the group $\mathrm{Mult}(\mathcal{L}_\mathcal{S})$ is the alternating group A_n.

This is the case for instance, if $n \in \{19, 25, 37, 43, 49, 61, 67, 73\}$.

$N:$

$+$	-1	0	1
-1	1	-1	0
0	-1	0	1
1	0	1	-1

$Q:$

$+$	$\bar{z}$	$\bar{y}$	$\bar{x}$	Ω	$-\bar{x}$	$-\bar{y}$	$-\bar{z}$
$\bar{z}$	$-\bar{z}$	$\bar{x}$	$\bar{y}$	$\bar{z}$	$-\bar{y}$	$-\bar{x}$	Ω
$\bar{y}$	$\bar{x}$	$-\bar{y}$	$\bar{z}$	$\bar{y}$	$-\bar{z}$	Ω	$-\bar{x}$
$\bar{x}$	$\bar{y}$	$\bar{z}$	$-\bar{x}$	$\bar{x}$	Ω	$-\bar{z}$	$-\bar{y}$
Ω	$\bar{z}$	$\bar{y}$	$\bar{x}$	Ω	$-\bar{x}$	$-\bar{y}$	$-\bar{z}$
$-\bar{x}$	$-\bar{y}$	$-\bar{z}$	Ω	$-\bar{x}$	$\bar{x}$	$\bar{z}$	$\bar{y}$
$-\bar{y}$	$-\bar{x}$	Ω	$-\bar{z}$	$-\bar{y}$	$\bar{z}$	$\bar{y}$	$\bar{x}$
$-\bar{z}$	Ω	$-\bar{x}$	$-\bar{y}$	$-\bar{z}$	$\bar{y}$	$\bar{x}$	$\bar{z}$

$$L: \begin{matrix} \Phi_{\bar{z},\bar{z}} & \Phi_{\bar{z},\bar{y}} & \Phi_{\bar{z},\bar{x}} & \Phi_{\bar{z},\Omega} & \Phi_{\bar{z},-\bar{x}} & \Phi_{\bar{z},-\bar{y}} & \Phi_{\bar{z},-\bar{z}} \\ \Phi_{\bar{y},\bar{z}} & \Phi_{\bar{y},\bar{y}} & \Phi_{\bar{y},\bar{x}} & \Phi_{\bar{y},\Omega} & \Phi_{\bar{y},-\bar{x}} & \Phi_{\bar{y},-\bar{y}} & \Phi_{\bar{y},-\bar{z}} \\ \Phi_{\bar{x},\bar{z}} & \Phi_{\bar{x},\bar{y}} & \Phi_{\bar{x},\bar{x}} & \Phi_{\bar{x},\Omega} & \Phi_{\bar{x},-\bar{x}} & \Phi_{\bar{x},-\bar{y}} & \Phi_{\bar{x},-\bar{z}} \\ \Phi_{\Omega,\bar{z}} & \Phi_{\Omega,\bar{y}} & \Phi_{\Omega,\bar{x}} & \Phi_{\Omega,\Omega} & \Phi_{\Omega,-\bar{x}} & \Phi_{\Omega,-\bar{y}} & \Phi_{\Omega,-\bar{z}} \\ \Phi_{-\bar{x},\bar{z}} & \Phi_{-\bar{x},\bar{y}} & \Phi_{-\bar{x},\bar{x}} & \Phi_{-\bar{x},\Omega} & \Phi_{-\bar{x},-\bar{x}} & \Phi_{-\bar{x},-\bar{y}} & \Phi_{-\bar{x},-\bar{z}} \\ \Phi_{-\bar{y},\bar{z}} & \Phi_{-\bar{y},\bar{y}} & \Phi_{-\bar{y},\bar{x}} & \Phi_{-\bar{y},\Omega} & \Phi_{-\bar{y},-\bar{x}} & \Phi_{-\bar{y},-\bar{y}} & \Phi_{-\bar{y},-\bar{z}} \\ \Phi_{-\bar{z},\bar{z}} & \Phi_{-\bar{z},\bar{y}} & \Phi_{-\bar{z},\bar{x}} & \Phi_{-\bar{z},\Omega} & \Phi_{-\bar{z},-\bar{x}} & \Phi_{-\bar{z},-\bar{y}} & \Phi_{-\bar{z},-\bar{z}} \end{matrix}$$

Choose a latin square on $\{-1,0,1\}$ for $\Phi_{\bar{z},\bar{z}}$. Using the corresponding *STS* blocks this determines the addition table $\Phi_{-\bar{z},-\bar{z}}$. Applying condition v) we obtain the places of the elements $(\Omega,0)$ in the table $\Phi_{\bar{z},-\bar{z}}$. We have two choices for $\Phi_{\bar{z},-\bar{z}}$ we could switch 1 and -1 there:

$\Phi_{\bar{z},\bar{z}}$	$\Phi_{\bar{z},-\bar{z}}$
$\Phi_{-\bar{z},\bar{z}}$	$\Phi_{-\bar{z},-\bar{z}}$

:

$+$	$(\bar{z},-1)$	$(\bar{z},0)$	$(\bar{z},1)$	$(-\bar{z},-1)$	$(-\bar{z},0)$	$(-\bar{z},1$
$(\bar{z},-1)$	$(-\bar{z},-1)$	$(-\bar{z},1)$	$(-\bar{z},0)$	$(\Omega,0)$	$(\Omega,1)$	$(\Omega,-$
$(\bar{z},0)$	$(-\bar{z},1)$	$(-\bar{z},0)$	$(-\bar{z},-1)$	$(\Omega,-1)$	$(\Omega,0)$	$(\Omega,1$
$(\bar{z},1)$	$(-\bar{z},0)$	$(-\bar{z},-1)$	$(-\bar{z},1)$	$(\Omega,1)$	$(\Omega,-1)$	$(\Omega,0$
$(-\bar{z},-1)$	$(\Omega,0)$	$(\Omega,-1)$	$(\Omega,1)$	$(\bar{z},-1)$	$(\bar{z},1)$	$(\bar{z},0$
$(-\bar{z},0)$	$(\Omega,1)$	$(\Omega,0)$	$(\Omega,-1)$	$(\bar{z},1)$	$(\bar{z},0)$	$(\bar{z},-$
$(-\bar{z},1)$	$(\Omega,-1)$	$(\Omega,1)$	$(\Omega,0)$	$(\bar{z},0)$	$(\bar{z},-1)$	$(\bar{z},1$

Since the set
$\{(-\bar{z},-1),(-\bar{z},0),(-\bar{z},1),(\bar{z},-1),(\bar{z},0),(\bar{z},1),(\Omega,0),(\Omega,1),(\Omega,-1)\}$ is closed under the loop operation it forms a *STS* subsystem with 9 elements. This determines the addition tables $\Phi_{\bar{z},\Omega}$ and $\Phi_{-\bar{z},\Omega}$.
Moreover, we do not have many choices for $\Phi_{\bar{z},\bar{z}}$ and $\Phi_{-\bar{z},-\bar{z}}$, because the only loop with 3 elements is the cyclic group, thus, for each of $\Phi_{\bar{z},\bar{z}}$ and $\Phi_{-\bar{z},-\bar{z}}$, we could only switch 1 and -1 in the addition table of N.
The reader can produce in the same way the addition tables corresponding to the cosets $\bar{y}$ and $-\bar{y}$, and to the cosets $\bar{x}$ and $-\bar{x}$, which are subsystems, as well.
Now we can freely choose a latin square on $\{-1,0,1\}$ for $\Phi_{\bar{z},\bar{y}}$ and, using the fact that $\bar{z}+\bar{y}=\bar{x}$, determine firstly the corresponding addition table and, consequently, determine the corresponding latin squares $\Phi_{\bar{z},-\bar{x}}$ and $\Phi_{\bar{y},-\bar{x}}$, according to Definition 6 iv):

$\Phi_{\bar{z},\bar{y}}$:

$+$	$(\bar{y},-1)$	$(\bar{y},0)$	$(\bar{y},1)$
$(\bar{z},-1)$	$(\bar{x},1)$	$(\bar{x},-1)$	$(\bar{x},0)$
$(\bar{z},0)$	$(\bar{x},0)$	$(\bar{x},1)$	$(\bar{x},-1)$
$(\bar{z},1)$	$(\bar{x},-1)$	$(\bar{x},0)$	$(\bar{x},1)$

,

(notice, in passing, that the latin square on $\{-1,0,1\}$ chosen here does not correspond to a group, nor a loop, with 3 elements, because it is not symmetric), hence

$\Phi_{\bar{z},-\bar{x}}$:

$+$	$(-\bar{x},-1)$	$(-\bar{x},0)$	$(-\bar{x},1)$
$(\bar{z},-1)$	$(-\bar{y},1)$	$(-\bar{y},-1)$	$(-\bar{y},0)$
$(\bar{z},0)$	$(-\bar{y},0)$	$(-\bar{y},1)$	$(-\bar{y},-1)$
$(\bar{z},1)$	$(-\bar{y},-1)$	$(-\bar{y},0)$	$(-\bar{y},1)$

,

$\Phi_{\bar{y},-\bar{x}}$:

$+$	$(-\bar{x},-1)$	$(-\bar{x},0)$	$(-\bar{x},1)$
$(\bar{y},-1)$	$(-\bar{z},-1)$	$(-\bar{z},0)$	$(-\bar{z},1)$
$(\bar{y},0)$	$(-\bar{z},0)$	$(-\bar{z},1)$	$(-\bar{z},-1)$
$(\bar{y},1)$	$(-\bar{z},1)$	$(-\bar{z},-1)$	$(-\bar{z},0)$

.