On marked Conway algebras and their invariants

Seonmi Choi
Joint with Yongju Bae and Seongjeong Kim

Kyungpook National University, Korea

LOOPS 2019 Conference

July 8, 2019

Contents

History

- (1984) V. F. R. Jones introduced a new polynomial for knots, which is called the *Jones polynomial* with the skein relation;

$$tV_{L_+}(t) - t^{-1}V_{L_-}(t) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L_0}(t) = 0.$$

- (1984.09~10.) The four research groups submitted the same result -the existence and properties of a new polynomial invariant for knots and links- , independently. This polynomial invariant generalizes both the Alexander-Conway polynomial and the Jones polynomials, which is called *HOMFLY polynomial* by combining the initials of its co-discoverers: J. Hoste, A. Ocneanu, K. Millet, P. J. Freyd, W. B. R. Lickorish, and D. N. Yetter. The skein relation is as follows.

$$vP_{L_+}(v, z) - v^{-1}P_{L_-}(v, z) = zP_{L_0}(v, z).$$

History

- (1985.04,1987) J. H. Przytycki and P. Traczyk introduced a *Conway algebra* and construct invariants of links valued in a Conway algebra, called *Conway type invariant*. The HOMFLY polynomial is obtained from Conway type invariants.
- The HOMFLY polynomial is also called the *HOMFLY-PT polynomial*. The addition of PT recognizes independent work carried out by J. H. Przytycki and P. Traczyk.
- (2017) L. H. Kauffman and S. Lambropoulou defined a new polynomial invariant with two different relations of self-crossings and mixed crossings.
- (2017) S. Kim generalized Conway algebras by applying two skein relations related to self-crossings and mixed crossings, which are called *generalized Conway algebras*.

Motivation

Link

HOMFLY-PT polynomial — Conway Algebra [Przytyski-Traczyk]

Polynomial inv. defined by Kauffman-Lambropoulou — Generalized Conway Algebra [Kim]

Surface-link

Polynomial inv. defined by Joung-Kamada-Kawauchi-Lee — ?

Marked Conway Algebra [Bae-C-Kim]

Generalized marked Conway Algebra

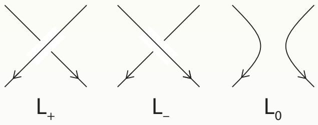
HOMFLY polynomial

Definition (Hoste-Ocneanu-Millet-Freyd-Lickorish-Yetter, 1985)

There is a unique function P from the set of isotopy classes of oriented links to the set of Laurent polynomials in v, z such that

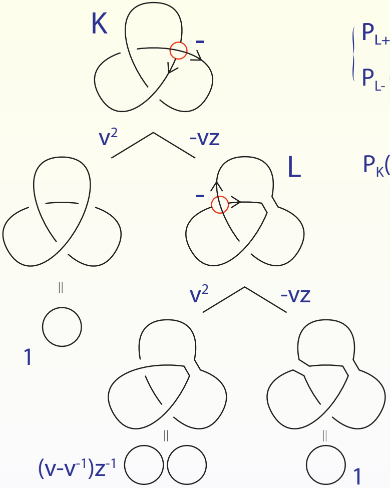
- $vP_{L_+}(v,z) - v^{-1}P_{L_-}(v,z) = zP_{L_0}(v,z)$,
- $P_L(v,z) = 1$ if L is the unknot.

The value $P(L)$ of a link L is called the *HOMFLY polynomial* of L.



HOMFLY polynomial of 3_1

- The HOMFLY polynomial P_K of the trefoil knot K is $2v^2 - v^4 + v^2z^2$.



$$\begin{cases} P_{L+}(v,z)=v^{-2}P_{L-}(v,z)+v^{-1}zP_{Lo}(v,z) \\ P_{L-}(v,z)=v^{2}P_{L+}(v,z)-v^{-1}zP_{Lo}(v,z) \end{cases}$$

$$\begin{aligned} P_K(v,z) &= v^2P_O(v,z)-vzP_L(v,z) \\ &= v^2P_O-vz[v^2P_{OO}-vzP_O] \\ &= v^2-v^3(v-v^{-1})+v^2z^2 \\ &= 2v^2-v^4+v^2z^2 \end{aligned}$$

Conway algebra

Definition (Przytyski-Traczyk, 1987)

Let $\mathcal{A}$ be a set with two binary operations $\circ$ and $/$ on $\mathcal{A}$. Let $\{a_n\}_{n=1}^{\infty} \subset \mathcal{A}$. A *Conway algebra* is the quadruple $(\mathcal{A}, \circ, /, \{a_n\}_{n=1}^{\infty})$ satisfying the following conditions:

1. $(a \circ b)/b = a = (a/b) \circ b$ for all $a, b \in \mathcal{A}$,
2. $a_n = a_n \circ a_{n+1}$ for all $n \in \mathbb{N}$,
3. $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$ for all $a, b, c, d \in \mathcal{A}$.

- The operation $/$ is the inverse operation of $\circ$.

Conway type invariant

- Let $\mathcal{L}$ be the set of equivalence classes of oriented link diagrams.

Proposition (Przytyski-Traczyk, 1987)

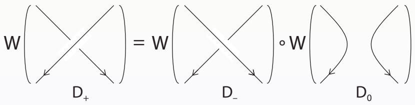
Let $(\mathcal{A}, \circ, /, \{a_n\}_{n=1}^{\infty})$ *be a Conway algebra. There exists an invariant* $W : \mathcal{L} \to \mathcal{A}$ *satisfying the following properties:*

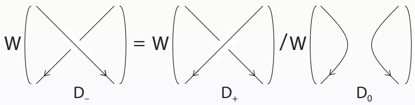
1. *For the trivial link* T_n *of* n *components,* $W(T_n) = a_n$*,*
2. *For each crossing,*

$$W(D_+) = W(D_-) \circ W(D_0),$$

where (D_+, D_-, D_0) *is the oriented Conway triple.*

The invariant W is called the Conway type invariant on the Conway algebra





Conway algebra

Example

1. Let $\mathcal{A} = \mathbb{Z}[p^{\pm 1}, q^{\pm 1}, r^{\pm 1}]$. Define binary operations $\circ$ and $/$ by

$$a \circ b = pa + qb + r \quad \text{and} \quad a/b = p^{-1}a - p^{-1}qb - p^{-1}r.$$

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined by the recurrence formula

$$a_1 = 1 \text{ and } (1-p)a_n = qa_{n+1} + r.$$

Then $(\mathbb{Z}[p^{\pm 1}, q^{\pm 1}, r^{\pm 1}], \circ, /, \{a_n\}_{n=1}^{\infty})$ is a Conway algebra.

- By substituting $p = v^{-2}$, $q = v^{-1}z$ and $r = 0$, one can obtain the next example.

Example

2. Let $\mathcal{A}' = \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. Define the binary operations $\circ$ and $/$ by

$$a \circ b = v^{-2}a + v^{-1}zb \quad \text{and} \quad a/b = v^2 a - vzb.$$

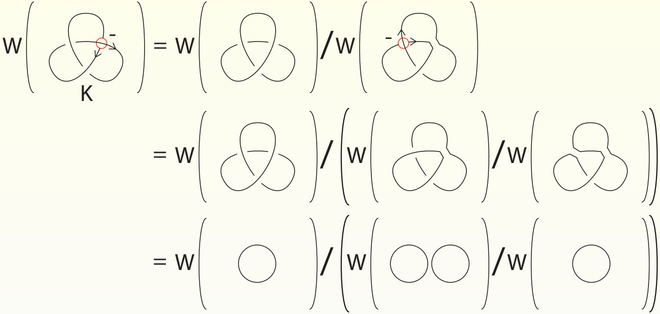
Denote $a_n = ((v - v^{-1})/z)^{n-1}$ for each n.
Then $(\mathbb{Z}[v^{\pm 1}, z^{\pm 1}], \circ, /, \{a_n\}_{n=1}^{\infty})$ is a Conway algebra.

- Then one can obtain the Conway type invariant W valued in the Conway algebra $\mathcal{A}'$. The Conway type invariant $W(L)$ of a link L is the HOMFLY polynomial of L.

- Hence we call it the HOMFLY-PT polynomial.

Conway type invariant of 3_1

- The calculation for the Conway type invariant of the trefoil K is $a_1/(a_2/a_1)$.



$= a_1/(a_2/a_1)$

- $a_1/(a_2/a_1) = a_1/(p^{-1}a_2 - p^{-1}qa_1 - p^{-1}r)$
  $= p^{-1}a_1 - p^{-1}q(p^{-1}a_2 - p^{-1}qa_1 - p^{-1}r) - p^{-1}r$
  $= 2p^{-1} + p^{-2}q^2 - p^{-2} + (p^{-2} + p^{-2}q - p^{-1})r$
- $2p^{-1} + p^{-2}q^2 - p^{-2} + (p^{-2} + p^{-2}q - p^{-1})r = 2v^2 - v^4 + v^2z^2$

Generalized Conway algebras

Definition (Kim, 2017)

Let $\widehat{\mathcal{A}}$ be a set with binary operations $\circ, *, /$ and $//$ on $\widehat{\mathcal{A}}$. Let $\{a_n\}_{n=1}^{\infty} \subset \widehat{\mathcal{A}}$. A *generalized Conway algebra* is the 6-tuple $(\widehat{\mathcal{A}}, \circ, *, /, //, \{a_n\}_{n=1}^{\infty})$ such that for $a, b, c, d \in \widehat{\mathcal{A}}$,

1. $(a \circ b)/b = (a/b) \circ b = a = (a * b)//b = (a//b) * b$,
2. $a_n = a_n \circ a_{n+1}$ for each $n \in \mathbb{N}$,
3. $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$,
4. $(a * b) * (c * d) = (a * c) * (b * d)$,
5. $(a \circ b) \circ (c * d) = (a \circ c) \circ (b * d)$,
6. $(a * b) * (c \circ d) = (a * c) * (b \circ d)$,
7. $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$.

Invariants of Generalized Conway algebras

Proposition (Kim, 2017)

Let $(\widehat{\mathcal{A}}, \circ, /, *, //, \{a_n\}_{n=1}^{\infty})$ *be a generalized Conway algebra. There uniquely exists the invariant* $\widehat{W} : \mathcal{L} \to \widehat{\mathcal{A}}$ *satisfying the following rules:*

1. *For each self crossing* c*,*

$$\widehat{W}(D_+^c) = \widehat{W}(D_-^c) \circ \widehat{W}(D_0^c).$$

2. *For each mixed crossing* c*,*

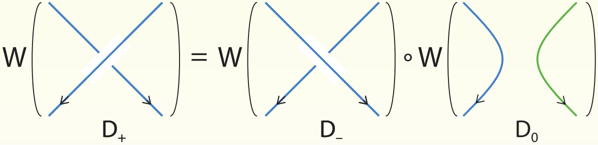
$$\widehat{W}(D_+^c) = \widehat{W}(D_-^c) * \widehat{W}(D_0^c).$$

3. *For the n-component trivial link* T_n*,* $\widehat{W}(T_n) = a_n$*.*

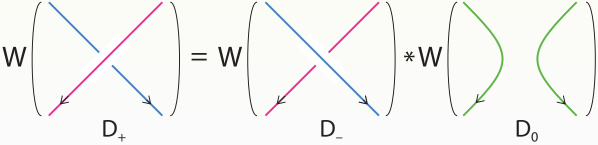
We call $\widehat{W}$ *the generalized Conway type invariant valued in* $(\widehat{\mathcal{A}}, \circ, /, *, //, \{a_n\}_{n=1}^{\infty})$*.*

Invariants of Generalized Conway algebras

1. For each self crossing c, $\widehat{W}(D^c_+) = \widehat{W}(D^c_-) \circ \widehat{W}(D^c_0)$.



2. For each mixed crossing c, $\widehat{W}(D^c_+) = \widehat{W}(D^c_-) * \widehat{W}(D^c_0)$.



Example of generalized Conway algebras and invariants

Example

- Let $\widehat{\mathcal{A}} = \mathbb{Z}[p^{\pm 1}, q^{\pm 1}, r]$. Define the binary operations $\circ, *, /$ and $//$ by

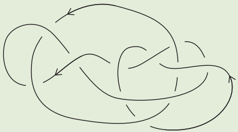
$$a \circ b = pa + qb, \quad a/b = p^{-1}a - p^{-1}qb,$$
$$a * b = pa + rb, \quad a//b = p^{-1}a - p^{-1}rb.$$

  Denote $a_n = (\frac{1-p}{q})^{n-1}$ for each n. Then $(\widehat{\mathcal{A}}, \circ, /, *, //, \{a_n\}_{n=1}^{\infty})$ is a generalized Conway algebra.

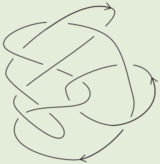
- By substituting $p = v^{-2}$ and $q = r = v^{-1}z$, we can obtain another generalized Conway algebra $(\mathbb{Z}[v^{\pm 1}, z^{\pm 1}], \circ, /, \{a_n\}_{n=1}^{\infty})$ and hence the generalized Conway type invariant valued in $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ is the HOMFLY-PT polynomial invariant.

Example (Kim, 2018)

It is well-known that two link $L_1 = L11n4180,0$ and $L_2 = L11n3580,1$ have the same HOMFLY-PT polynomial. There exists a generalized Conway algebra $\widehat{\mathcal{A}}$ such that $\widehat{W}(L_1) \neq \widehat{W}(L_2)$.



L11n418{0,0}

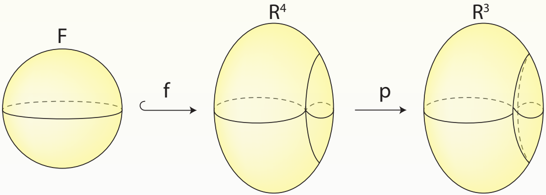


L11n358{0,1}

Surface-links

Definition

- A *surface-link* is the image of a smooth embedding of a closed surface F in $\mathbb{R}^4$.
- A surface-link is called a *surface-knot*, if F is connected.
- A surface-link is said to be *orientable* if F is orientable.



Marked graphs

Definition

- A *marked graph* is a 4-valent graph embedded in $\mathbb{R}^3$ with marked vertices.
- A *marked graph diagram* can be described by a diagram in $\mathbb{R}^2$, which is a link diagram with some 4-valent vertices equipped with *markers*.

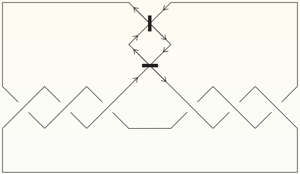
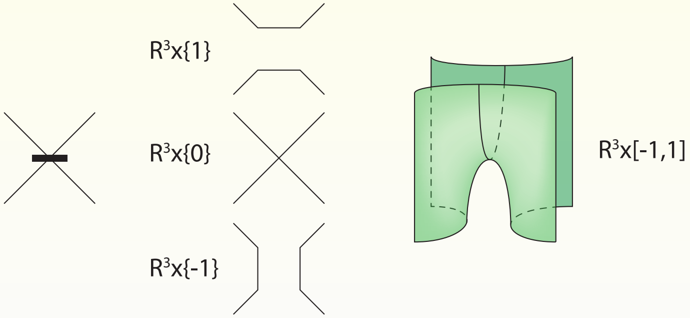


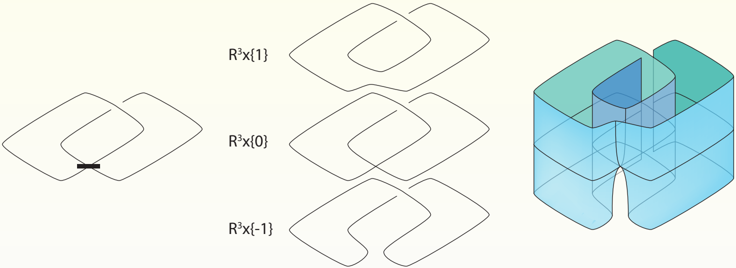
Figure: The spun trefoil

Surface-links and marked graph diagrams



Surface-links and marked graph diagrams

Prop. For an admissible marked graph diagram D, we can construct a surface-link $F(D)$ from D.



Prop. Every surface-link F can be represented by an admissible marked graph diagram D.

Equivalent oriented marked graphs

- {oriented marked graphs}/ $\cong$
  $\leftrightarrow$ {oriented marked graph diagrams}/$\Gamma_1, \cdots, \Gamma_5$

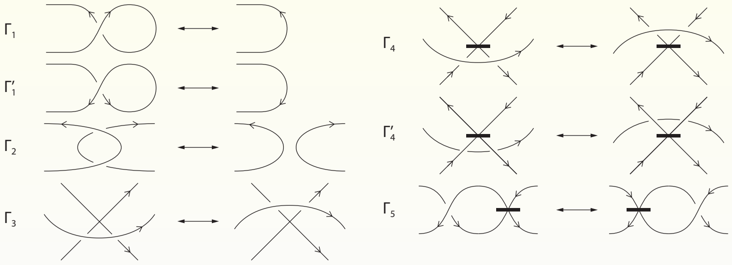


Figure: Yoshikawa moves of type 1

Equivalent oriented surface-links

- {oriented surface-links}/ $\cong$
  $\leftrightarrow$ {oriented marked graph diagrams}/$\Gamma_1, \cdots, \Gamma_8$

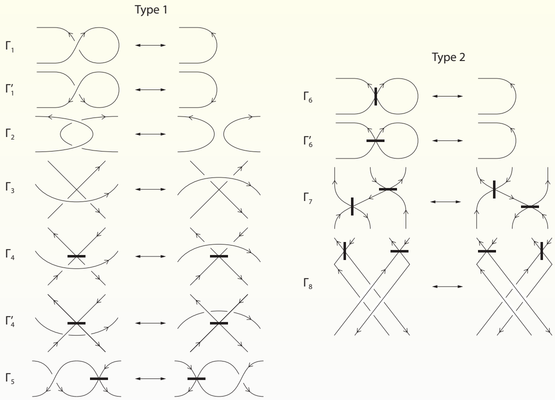


Figure: Yoshikawa moves

Polynomial for oriented marked graphs

Proposition (Joung-Kamada-Kawauchi-Lee, 2017)

The polynomial $\ll \cdot \gg$ is an invariant for oriented marked graphs, i.e. for an oriented marked graph diagram D, the polynomial $\ll D \gg = \ll D \gg (a, x, y)$ is invariant under Yoshikawa moves of type 1. Moreover,

- $\ll O \gg = 1$.
- $\ll D \sqcup O \gg = (a^{-6} + 1 + a^{6}) \ll D \gg$ *for any oriented link diagram* D.
- *For any skein triple* (D_+, D_-, D_0) *of link diagrams,*

$$a^{-9} \ll D_+ \gg - a^{9} \ll D_- \gg = (a^{-3} - a^{3}) \ll D_0 \gg .$$

- *For any marked skein triple* (D^m, D^m_0, D^m_∞),

$$\ll D^m \gg = x \ll D^m_\infty \gg + y \ll D^m_0 \gg .$$

Polynomial for oriented marked graphs

Proposition (cont')

- *For any skein triple* (D_+, D_-, D_0) *of link diagrams,*

$$a^{-9} \ll D_+ \gg -a^9 \ll D_- \gg = (a^{-3} - a^3) \ll D_0 \gg .$$

- *For any marked skein triple* (D^m, D^m_0, D^m_∞),

$$\ll D^m \gg = x \ll D^m_\infty \gg + y \ll D^m_0 \gg .$$

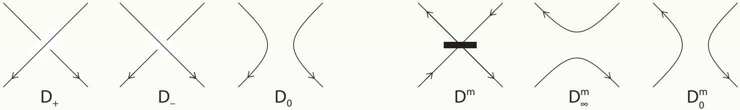


Figure: Skein triple and Marked Skein triple

Polynomial for oriented surface-links

Proposition (Joung-Kamada-Kawauchi-Lee, 2017)

Let L be an oriented surface-link and let D be an oriented marked graph diagram presenting L. Then the polynomial $\ll D \gg \in \mathbb{Z}[a^{\pm 1}, x, y]$, modulo the ideal generated by $\triangle(a)$, is an invariant of L up to multiplication by powers of $(a^{-6}+1+a^6)x+y$ and $x+(a^{-6}+1+a^6)y$ where $\triangle(a)=(a^{-6}+1+a^6)^2-1$.

- In this talk, this polynomial is called a JKKL polynomial.

Marked Conway algebra

Definition (Bae-C-Kim, 2018)

Let $\mathcal{MA}$ be a set with three binary operations $\circ, /$ and $\bullet$ on $\mathcal{MA}$. Let $\{a_n\}_{n=1}^{\infty} \subset \mathcal{MA}$. A *marked Conway algebra* is the 5-tuple $(\mathcal{MA}, \circ, /, \bullet, \{a_n\}_{n=1}^{\infty})$ satisfying

1. $(a \circ b)/b = a = (a/b) \circ b$ for $a, b \in \mathcal{MA}$,
2. $a_n = a_n \circ a_{n+1}$ for every $n \in \mathbb{N}$,
3. $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$ for $a, b, c, d \in \mathcal{MA}$,
4. $(a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d)$ for $a, b, c, d \in \mathcal{MA}$.

Marked Conway algebra

Example

Let $\mathcal{MA} = \mathbb{Z}[p^{\pm 1}, q^{\pm 1}, z^{\pm 1}, x, y]$. Define binary operations $\circ$, $/$ and $\bullet$ by

$$a \circ b = pa + qb + z,$$

$$a/b = p^{-1}a - p^{-1}qb - p^{-1}z,$$

$$a \bullet b = xa + yb.$$

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined by the recurrence formula

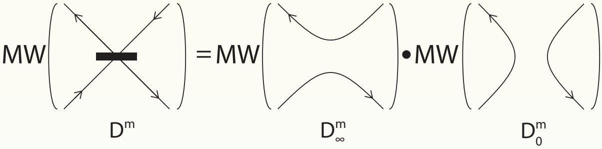
$$a_1 = 1, \quad (1-p)a_n = qa_{n+1} + z.$$

Then $\left(\mathbb{Z}[p^{\pm 1}, q^{\pm 1}, z^{\pm 1}, x, y], \circ, /, \bullet, \{a_n\}_{n=1}^{\infty}\right)$ is a marked Conway algebra.

Construct the marked Conway type invariant

Now we construct the invariant MW of marked graph diagrams modulo Yoshikawa moves of type1 valued in a marked Conway algebra $\mathcal{MA}$.
If a marked graph diagram D has markers, then for each marker m, we splice m satisfying

$$MW(D^m) = MW(D^m_\infty) \bullet MW(D^m_0). \tag{1}$$



If D has no markers, then

$$MW(D) = W(D).$$

Construct the marked Conway type invariant

Theorem (Bae-C-Kim, 2018)

Let $\mathcal{ML}_I$ be the set of equivalence classes of oriented marked graph diagrams modulo Yoshikawa moves of type 1. Let $(\mathcal{MA}, \circ, /, \bullet, \{a_n\}_{n=1}^{\infty})$ be a marked Conway algebra. Then there uniquely exists the invariant $MW : \mathcal{ML}_I \to \mathcal{MA}$, satisfying the following rules:

1. *For a marker m the following relation holds:*

$$MW(D^m) = MW(D^m_\infty) \bullet MW(D^m_0).$$

2. *If D has no markers, then*

$$MW(D) = W(D).$$

The invariant MW is called the marked Conway type invariant valued in $\mathcal{MA}$.

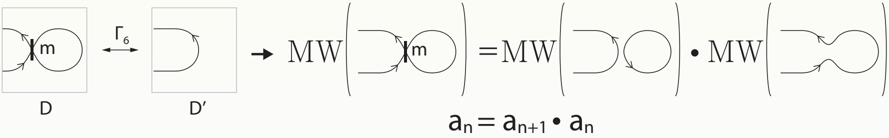
Marked Conway type invariant

Lemma

Let $(\mathcal{MA}, \circ, /, \bullet, \{a_n\}_{n=1}^{\infty})$ *be a marked Conway algebra such that for all* $n \in \mathbb{N}$ *and for every* $a, b, c, d \in \mathcal{MA}$,

$$a_n \bullet a_{n+1} = a_n = a_{n+1} \bullet a_n \text{ and } (a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d).$$

Then the marked Conway type invariant $MW : \mathcal{ML}_I \to \mathcal{MA}$ *is invariant under* Γ_6 *and* Γ_6'.



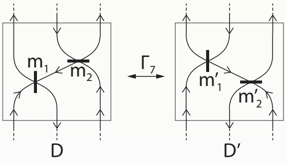
Marked Conway type invariant

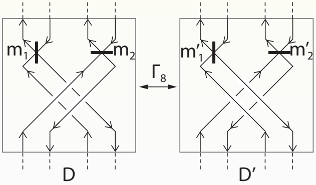
Lemma

Let $(\mathcal{MA}, \circ, /, \bullet, \{a_n\}_{n=1}^{\infty})$ *be a marked Conway algebra such that for all* $n \in \mathbb{N}$ *and for all* $a, b, c, d \in \mathcal{MA}$,

$$a_n = a_{n+2} \text{ and } (a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d).$$

Then the marked Conway type invariant $MW : \mathcal{ML}_I \to \mathcal{MA}$ *is invariant under* Γ_7 *and* Γ_8.





Marked Conway type invariant

Theorem (Bae-C-Kim, 2018)

Let $\mathcal{ML}_{II}$ be the set of equivalence classes of oriented marked graph diagrams modulo oriented Yoshikawa moves. Let $(\mathcal{MA}, \circ, /, \bullet, \{a_n\}_{n=1}^{\infty})$ be a marked Conway algebra with three conditions: for all $n \in \mathbb{N}$ and for all $a, b, c, d \in \mathcal{MA}$,

$$a_n \bullet a_{n+1} = a_n = a_{n+1} \bullet a_n, \qquad a_n = a_{n+2},$$

$$(a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d).$$

Then there uniquely exists the invariant of oriented surface-links, $MW : \mathcal{ML}_{II} \to \mathcal{MA}$, satisfying the following conditions:

1. *For a marker m, $MW(D^m) = MW(D^m_\infty) \bullet MW(D^m_0)$,*
2. *For a classical crossing c, $MW(D^c_+) = MW(D^m_-) \circ MW(D^m_0)$,*
3. *If D has no markers, then $MW(D) = W(D)$.*

The invariant MW is the marked Conway type invariant valued in $\mathcal{MA}$ for oriented surface-links.

Marked Conway algebra

Example

Let $\mathcal{MA} = \mathbb{Z}[p^{\pm 1}, q^{\pm 1}, x, y]$. Define binary operations $\circ$, $/$ and $\bullet$ by

$$a \circ b = pa + qb, \quad a/b = p^{-1}a - p^{-1}qb, \quad a \bullet b = xa + yb.$$

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined by the recurrence formula

$$a_1 = 1, \quad (1-p)a_n = qa_{n+1}.$$

- $\left(\mathbb{Z}[p^{\pm 1}, q^{\pm 1}, x, y]/(x + y(1-p)q^{-1}, y + x(1-p)q^{-1}), \circ, /, \bullet, \{a_n\}_{n=1}^{\infty}\right)$ is a marked Conway algebra satisfying $a_n \bullet a_{n+1} = a_n = a_{n+1} \bullet a_n$.
- $\left(\mathbb{Z}[p^{\pm 1}, q^{\pm 1}, x, y]/((1-p)^2 q^{-2}), \circ, /, \bullet, \{a_n\}_{n=1}^{\infty}\right)$ is a marked Conway algebra satisfying $a_n = a_{n+2}$.

Marked Conway type invariant

Example

Let $\mathcal{MA} = \mathbb{Z}[a^{\pm 1}, x, y]$. Define binary operations $\circ$, $/$ and $\bullet$ by

$$\alpha \circ \beta = a^{18}\alpha + (a^6 - a^{12})\beta,$$

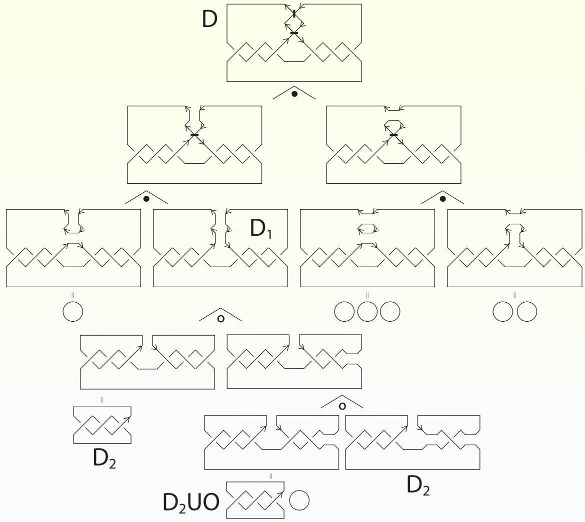
$$\alpha / \beta = a^{-18}\alpha + (a^{-6} - a^{-12})\beta,$$

$$\alpha \bullet \beta = x\alpha + y\beta.$$

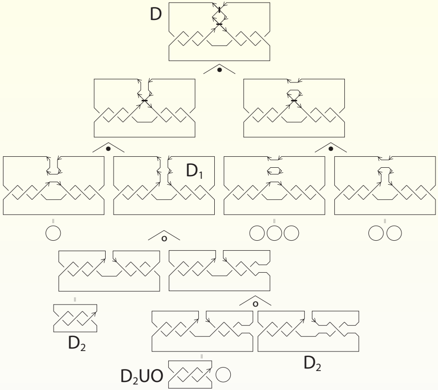
Denote $a_n = (a^{-6} + 1 + a^6)^{n-1}$, for each $n \in \mathbb{N}$.

- This is obtained from the marked Conway algebra in the previous example by substituting $p = a^{18}$ and $q = (a^6 - a^{12})$.
- Moreover, the JKKL polynomial invariant is obtained from the marked Conway type invariant MW valued in $\mathcal{MA}$.

Calculation of marked Conway type invariants



Calculation of marked Conway type invariants



- $MW(D) = MW(8_1) = [a_2 \bullet MW(D_1)] \bullet [a_3 \bullet a_2]$
- $MW(D_1) = MW(D_2) \circ [MW(D_2 \sqcup O) \circ MW(D_2)]$
  $= [a_1/(a_2/a_1)] \circ [(a_2/(a_3/a_2)) \circ (a_1/(a_2/a_1))]$

Generalized marked Conway algebra

Definition

Let $\widehat{\mathcal{MA}}$ be a set with five binary operations $\circ, *, /, //$ and $\bullet$ on $\widehat{\mathcal{MA}}$. Let $\{a_n\}_{n=1}^{\infty} \subset \widehat{\mathcal{MA}}$. A *generalized marked Conway algebra* is the 7-tuple $(\widehat{\mathcal{MA}}, \circ, *, /, //, \bullet, \{a_n\}_{n=1}^{\infty})$ such that for $a, b, c, d \in \widehat{\mathcal{MA}}$,

1. $(a \circ b)/b = (a/b) \circ b = a = (a * b)//b = (a//b) * b$,
2. $a_n = a_n \circ a_{n+1}$ for each $n \in \mathbb{N}$,
3. $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$,
4. $(a * b) * (c * d) = (a * c) * (b * d)$,
5. $(a \circ b) \circ (c * d) = (a \circ c) \circ (b * d)$,
6. $(a * b) * (c \circ d) = (a * c) * (b \circ d)$,
7. $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$,
8. $(a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d)$.

Construct a generalized marked Conway type invariant

Theorem (Bae-C-Kim, 2018)

Let $\mathcal{ML}_I$ be the set of equivalence classes of oriented marked graph diagrams modulo Yoshikawa moves of type1. For a generalized marked Conway algebra $(\mathcal{MA}, \circ, *, /, //, \bullet, \{a_n\}_{n=1}^{\infty})$, there uniquely exists the invariant $\widehat{MW} : \mathcal{ML}_I \to \mathcal{MA}$, satisfying the following rules:

1. *For a marker m the following relation holds:*

$$\widehat{MW}(L^m) = \widehat{MW}(L^m_\infty) \bullet \widehat{MW}(L^m_0). \tag{2}$$

2. *If L has no markers, then*

$$\widehat{MW}(L) = \widehat{W}(L) \tag{3}$$

   where $\widehat{W}$ is a generalized Conway type invariant.

The invariant $\widehat{MW}$ is an invariant for oriented marked graphs.

Thank you for your attention.