Introduction to Algebra 2 **Final Exam** **12 June 2023**

1. Define the following concepts: *Linear map, linear transformation, kernel, image, isomorphism.* State the homomorphism theorem (dimension theorem). Desrcibe the concept of projections relating the subspaces $\mathrm{Im}(\pi)$ and $\mathrm{Ker}(\pi)$.

2. Given $A \in M_n(F)$ what are its eigenvalues, eigevectors and eigenspaces? Define the minimal polynomial and the characteristic polynomial of A. What are the relations between the concepts mentioned so far? State the Cayley-Hamilton Theorem. Construct a 3×3 complex matrix with two eigenvalues and a degree 3 minimal polynomial.

3. Define the real and complex Euclidean spaces and discuss their differences. State the theorems about their geometry. Does the following matrix of a real bilinear function define a Euclidean inner product?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

4. Define when a matrix is orthogonal/unitary and give equivalent descriptions. Define the concept of orthogonal/unitary diagonalisation. State the theorems about orthogonal diagonalisability and about unitary diagonalisability. Give an example of a 2×2 real matrix that cannot be unitarily diagonalised.

5. What is the pseudoinverse of a $k \times n$ (real) matrix A? Relate it to the row/column/null spaces of A. Describe how it can be used in "solving" linear equations. How can the pseudoinverse be computed (using full-rank factorisations)? State the Moore-Penrose theorem.

6. Give at least two types of dynamical problems that lead to assessing non-negative matrices. Define the spectrum and spectral radius of a matrix and irreducibility. State Perron's Lemma and the Perron-Frobenius Theorem. Give the stationary distribution of a Markov chain with the following transition matrix $(0 < 1 - a < b < 1)$

$$T = \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}.$$